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DISTRIBUTION-FREE DETECTION PROCEDURES

J. C. Hancock, Principal Investigator
D. G. Lainiotis

Aeronautical Systems Division
Air Force Systems Command
Wright-Patterson Air Force Base

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FOREWORD

This report was prepared by Purdue University under USAF Contract Number AF 33(657)-10709. The contract was initiated under Project 4335, "(U) Applied Communications Research for AF Vehicles," Task 433529, "Basic Techniques and Systems Integration." The work was administered under the direction of the Communications Branch, Electronic Warfare Division, Air Force Avionics Laboratory, Wright-Patterson Air Force Base, Ohio. Mr. B. W. Russell was project engineer.

Dr. John Hancock, Purdue University, was the Principal Investigator on the contract. This report covers work conducted from February 1963 to December 1964.

The first volume in this series is a tutorial overview of the several problem areas investigated under this effort and serves to integrate and place in perspective the more detailed analysis presented in succeeding volumes. The relation of this Volume IV, "Distribution-Free Detection Procedures," to the overall program may be obtained by referring to Volume I.

Throughout the course of this work, the principal investigators have benefited from the several discussions with Mr. Blinn W. Russell, the project monitor, and his associates. His interest in this work is gratefully acknowledged.

ABSTRACT

A class of two-input detection systems for digital communication over random and unknown channels is investigated. The systems investigated possess false-alarm rates which are invariant for wide classes of channel statistics.

Specifically, coincidence detection procedures with invariant or distribution-free false-alarm rates are proposed and investigated. The only information concerning the channel statistics which is required by these detectors is the median of the noise under no-signal conditions. The coincidence procedures are subsequently modified so that the detectors utilizing them become either learning systems with respect to slowly time-varying and/or unknown location parameters or adaptive systems with respect to rapidly varying and/or unknown location parameters. The classes of detection problems for which the false-alarm rates of the above procedures detectors remain distribution-free are also obtained.

In addition to the distribution-free coincidence detectors, a detector based on the T-statistic, and well suited for the detection of stochastic signals in noise, is proposed and investigated. The T-statistic is then modified so that the detector utilizing it becomes an adaptive system with respect to rapidly varying and/or unknown location parameters. The wide classes of detection problems for which the above detectors remain distribution-free are also obtained.

The distribution-free detectors are then applied to various practical detection problems, and their performances are evaluated and compared to the performances of comparable likelihood detectors.

This technical documentary report has been reviewed and is approved.

LIONEL W. ROBERTS
Lt Colonel, USAF
Chief, Electronic Warfare Division

TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF ILLUSTRATIONS

LIST OF IMPORTANT SYMBOLS

Symbol	Definition	Page	Chapter
$N(t)$	Reference sample function	13	2
$Y(t)$	Data sample function	13	2
m	Number of samples from $N(t)$	13	2
n	Number of samples from $Y(t)$	13	2
Y_i	i-th sample from $Y(t)$	13	2
X_i	i-th sample from $N(t)$	13	2
H_o	Hypothesis that signal is absent	13	2
H_1	Hypothesis that signal is present	13	2
$F_o(y)$	Distribution function of X_i	13	2
$G_o(y)$	Distribution function of Y_i when signal is absent	13	2
$G_{\theta_i}(y)$	Distribution function of Y_i when signal is present	13	2
θ	Signal-to-noise ratio parameter	14	2
α	Time interval between independent samples	16	2
S_n	Distribution-free statistic	17	2
$E_\theta(S_n)$	Mean of distribution-free statistic	17	2
$E_\theta'(S_n)$	Derivative of $E_\theta(S_n)$ with respect to θ	17	2
$\sigma_\theta^{\;2}(S_n)$	Variance of the distribution-free statistic	17	2
K	Positive constant	17	2
α	Probability of false alarm	18	2
β	Probability of false dismissal	18	2
S_α	Decision threshold	18	2
erf^{-1}	Inverse of the error function	19	2
K	Efficiency of the test statistic	20	2
L_n	Test statistic	20	2
R	Information rate possible with S_n	20	2

SYMBOLS (continued)

Symbol	Definition	Page	Chapter
R^*	Information rate possible with L_n	20	2
$ARE_{U^*,U}$	Asymptotic relative efficiency of the detector U^* with respect to the detector U	21	2
n	Number of samples required by L_n	21	2
K^*	Efficacy of statistic L_n	21	2
$\frac{S}{N}$	Signal-to-noise ratio	21	2
$\sum_{i=1}^{n}$	Sum of n terms	24	3
y_i	i-th observation from Y(t)	24	3
x	Threshold level	24	3
c(z)	Function of z	24	3
$E_o[S_n]$	Mean of distribution-free statistic under no-signal conditions	24	3
$\sigma_o^2[S_n]$	Variance of distribution-free statistic under no-signal conditions	25	3
z_p	pth quantile of noise samples under no-signal conditions	26	3
M	Median of noise samples under no-signal conditions	27	3
$S_n(M)$	Test statistic with M as threshold	27	3
Z(t)	Result of subtracting M from Y(t)	28	3
t_i	i-th sampling instant	29	3
$\overline{\theta}$	Average signal-to-noise ratio	31	3
$\overline{\theta^2}$	Mean square value of signal-to-noise ratio	32	3
$f_o(y)$	Probability density function of X_i under no-signal conditions	34	3
L^*_n	Test statistic for the likelihood detector	34	3
Γ( )	Gamma function	35	3
c	Parameter of gaussian-plus-impulse noise distribution	35	3

SYMBOLS (continued)

Symbol	Definition	Page	Chapter
σ^2	Mean square value of noise stochastic process	36	3
σ_N^2	Mean square value of narrow-band white gaussian noise	39	3
P_E	Probability of error	41	3
$\hat{M}_1$	Estimate of unknown median	48	4
$F_z(z)$	Distribution function of $Z(t)$ under no-signal conditions	50	4
$g_o(y)$	Probability density function of Y_1	51	4
$f_M(x)$	Probability density function of $\hat{M}_1$	51	4
$f_{z_1}(z)$	Probability density function of Z_1	52	4
$p(\theta_1)$	Probability density function of θ_1	53	4
$K(\hat{M})$	Efficacy of $S_n(\hat{M})$	55	4
$\delta(x)$	Dirac delta function	55	4
erf	Error function	60	4
$f_{o_1}(x)$	Probability density function of X_1 under no-signal conditions	71	5
$F_{z_1}(z)$	Probability distribution function of Z_1	71	5
T_{mn}	T-statistic	86	6
$\psi(y_1, x_j)$	Function of y_1 and x_j	86	6
$E_0[T_{mn}]$	Mean of T_{mn} under no-signal conditions	87	6
$\sigma_o^2[T_{mn}]$	Variance of T_{mn} under no-signal conditions	88	6
M	Median of data sample under no-signal conditions	89	6
N	Median of reference sample under no-signal conditions	89	6
$T_{mn}(M,N)$	T-statistic for nonzero median samples	89	6
$E_\theta[T_{mn}(M,N)]$	Mean of $T_{mn}(M,N)$ when signal is present	90	6
$\sigma_\theta^2[T_{mn}]$	Variance of T_{mn} when signal is present	90	6
$K(M,N)$	Efficacy of $T_{mn}(M,N)$	91	6

SYMBOLS (continued)

Symbol	Definition	Page	Chapter
V_i	Difference between y_{2i} and y_{2i-1}	98	6
U_j	Difference between x_{2j} and x_{2j-1}	98	6
M_k	Median of Y_i	100	6
N_ℓ	Median of X_ℓ	100	6
$F_{y_k}(y)$	Distribution function of Y_k	100	6
$F_{x_\ell}(x)$	Distribution function of X_ℓ	100	6
$f_{v_i}(v)$	Probability density function of V_i	102	6
$G_v(x)$	Distribution function of V when signal is present	103	6
$F_v(x)$	Distribution function of V_i under no-signal conditions	103	6
$g_v(x)$	Probability density function of V_i under signal present conditions	104	6

Chapter 1

INTRODUCTION

1.1 General Background of Problem and Brief Review of the Literature

Systems for detecting the presence or absence of a signal in noise have been extensively investigated. Many of these investigations have been based on the assumption that information is available concerning the probability distributions of the noise only and of the mixture of signal and noise. The distributions are usually assumed to be gaussian, and the test statistic most often utilized is based on the likelihood ratio. In order for the latter to be computed, knowledge of the form of the distributions is required.

However, the statistics are not always known or readily obtainable. In many practical cases of interest - such as subsurface communications, underwater sound detection, communications under jamming conditions, and space communications - the probability distributions may not be known, nor can they be easily obtained. Extensive statistical studies, such as those by Pearson and Geyen (1,2), have been made of the sensitivity of various likelihood ratio tests to non-normality.

Since likelihood detectors are inapplicable whenever there is not information concerning the functional form of the underlying distributions, and since it is not possible to insure a specified value of a chosen index of performance when a likelihood detector is used and the distributions are not known, it is necessary to seek detection procedures that are invariant in some sense under a change of the underlying probability distributions - as indicated by Middleton (3). Such procedures may be based on a reference or control sample obtained under noise only conditions,

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Manuscript released by authors on February 1965 for publication as an RTD Technical Report which can be compared with a sample obtained when a decision is to be made on the presence or absence of signal. In other words, since the received data cannot be described by distribution functions of known form, two samples can be utilized - the reference sample obtained when it is known that only noise exists in the channel, and a data sample obtained under unknown conditions - the decision being based on comparison between the two samples. Since random processes are being dealt with, this comparison must be a comparison of statistical properties. The logic behind this approach is based on the assumption that the presence of a signal will cause a difference to exist between the statistics of the reference and data samples.

Capon, Groginsky, Rushforth, Hancock, Wolff and Kanefsky have all rather recently applied distribution-free statistical tests to the detection problem.

Capon (4,5,6) has applied many tests to the detection problem, such as the Mann-Whitney, the Wald-Wolfowitz, the Kolmogorov-Simirnov and the rank order tests. Of these, the only ones of interest to the communications engineer are the highly efficient Mann-Whitney and the rank order tests. However, in order for the rank order statistic to be applied to a specific detection problem, weighting factors must be known. The information on these weighting factors cannot be obtained unless the noise distribution is known and even then the factors are often difficult to compute. Thus, the rank order tests as treated by Capon are not truly distribution-free. The Mann-Whitney test, though, is distribution free.

Groginsky (7) proposed procedures whereby the weight function of the rank order detector is determined from the outcome of previous trial of the detector, thus effectively removing the necessity of knowledge concerning the distributions. The ability of such a detector to obtain the required weight function for a wide class of distribution functions, and to follow changes in the structure of the signals and noise, constitutes its adaptive feature. Stability and performance of various schemes to update the weight function were also studied.

Rushforth (8) re-examined rank order tests and obtained results of practical importance for detection problems. Hancock (9) re-examined the Mann-Whitney test and obtained results for non-gaussian noise statistics.

Wolff (10) pointed out that the polarity coincidence correlator is non-parametric in that the false-alarm rate depends only on the median of the noise. The latter was assumed to be zero. However, the assumption of zero median for the noise may not always be a realistic assumption, in which case this test is no longer distribution-free.

Kanefsky (11) subsequently showed that the effect of a non-zero median on the polarity coincidence correlator was removed when an adaptive procedure to set the threshold level for inputs with quasi-stationary medians - such as described by Eykhoff (12) and Zadeh (13) - was employed. The modified polarity coincidence correlator was then applied to various detection problems of practical importance and its performance compared to that of various optimum and suboptimum detectors.

From the citings of the above literature, it is seen that there have been numerous investigations of the applicability of various distribution-free tests to the detection problem. However, the previous investigations by no means exhaust the subject. There are many very promising distribution-free tests that have not been previously investigated from the detection theory standpoint. Some involve the use of coincidence detection procedures. These coincidence procedures for detecting the presence or absence of a signal in noise have been studied extensively by Harrington (14), Schwartz (15), Capon (16), and Bunimovich (17). The detectors used choose a threshold level and count the number of observations that exceed this level. On the basis of this number, the detector decides whether or not there is a signal present. In earlier investigations of coincidence procedures, the threshold was chosen, on the basis of intuition and engineering judgment, to be the mean of the input waveform under no-signal conditions. This choice of threshold leads to a suboptimum coincidence procedure. Later, an analytical and more sophisticated approach to the subject was taken, and optimum coincidence procedures were obtained for weak signals in noise. The optimum coincidence detection procedure chooses the threshold level in such a manner that it requires the minimum input signal-to-noise ratio to insure a specified information rate and error probability. Some investigators have obtained the optimum coincidence detector for particular detection problems by means of a point-by-point graphical procedure. However, it must be emphasized that to obtain the optimum threshold, complete knowledge of the first-order probability distributions under signal and under no-signal conditions is required. Moreover, the threshold is optimum only for the particular detection problem for which it was obtained.

In general, the coincidence detection procedures proposed in the past are parametric procedures - and hence inapplicable whenever the probability distributions are unknown. They are optimum for a particular detection problem for which they have been obtained and become suboptimum, if not useless, under different circumstances. The most important drawback of the parametric coincidence procedures in the face of unknown distributions is their inability to specify and guarantee the attainment of a desired value of a specified index of performance, such as false-alarm rate. Thus, a need exists for coincidence detection procedures that are applicable when the distributions are unknown and that possess an index of performance invariant under changes in the statistics of the detection problem.

1.2 General Assumptions

The cardinal assumptions on which this investigation is based must be emphasized. These are a) that two sample functions are available; and b) that independent samples can be obtained from the two sample functions without knowing the underlying probability distributions. In addition, all the investigations are restricted to the detection of weak signals in noise. Comparisons between nonparametric, or distribution-free detection procedures, and parametric ones are made on the basis of the concept of asymptotic relative efficiency which is a measure of relative information rate for specific error probability.

1.3 Methods and Procedures

In this investigation, two distribution-free test statistics are utilized; these are characterized by simplicity, lack of severe restrictive assumption, and high efficiency. They consist of the sign, or median, test statistic and the T-statistic. The first is well suited for the detection of deterministic signals in noise, while the T-statistic is well suited for the detection of stochastic signals in noise.

The sign test statistic as used here constitutes a coincidence detection procedure. In this investigation, coincidence detection procedures with invariant false-alarm rates for a wide class of distribution functions are proposed and investigated. Conditions under which the coincidence procedures remain distribution-free are also obtained. The detectors based on the distribution-free coincidence procedures are then applied to various detection problems of practical importance; their performance in the problems is evaluated and compared to that of likelihood detectors.

In the distribution-free coincidence procedures investigated here, the threshold is chosen so that the test statistic possesses, under no-signal conditions, a known distribution with constant and known mean and variance, independent of the statistics of the detection problem. The invariant nature of the test statistic distribution under no-signal conditions insures a false alarm rate invariant with respect to changes in the channel statistics. The threshold is chosen to be a specified noise distribution quantile, namely, the median (recalling that the "median" is the point at which the cumulative distribution is 0.5) - hence the name "median detector" for the distribution-free coincidence procedure. It must be noted that to employ the median detector, the median of the noise under no-signal conditions must be known. The latter is the only information concerning the channel statistics required by the median detector. Since, in many detection problems of interest, even this minimal information concerning the channel statistics may not be available, the median detector is made a learning system with respect to the unknown median for a wide class of distribution functions. Thus, a learning procedure is herein proposed and investigated whereby the threshold is adjusted to maintain an invariant false-alarm rate even in the case of an unknown, stationary or quasi-stationary noise median. The conditions under which the learning median detector remains distribution-free are obtained. The learning median detector is then applied to a gaussian and to a non-gaussian situation of practical importance, and its performance and learning efficiency are evaluated and compared to the performance and learning efficiency of likelihood detector.

The median detector is also made adaptive to rapid changes in the structure of the noise for a wide class of distribution functions; that is, an adaptive procedure is proposed and investigated whereby the threshold is adjusted to maintain an invariant false-alarm rate even for non-stationary noise medians. The conditions under which the adaptive median detector remains distribution-free are obtained. The adaptive median detector is then applied to various detection problems and its performance is evaluated and compared to that of other distribution-free detectors and to that of likelihood detectors.

In this investigation, the T-statistic, an efficient test statistic for the detection of changes in variance, is applied to the problem of detecting stochastic signals in noise. The conditions under which the T-statistic remains distribution-free are obtained. To employ the T-detector based on the T-statistic, the medians of the noise in the reference and data channels must be known. The T-detector is applied to a detection problem of practical importance, and its performance in the problem is evaluated and compared to the performance of the equivalent likelihood detector.

The T-detector is also made adaptive to changes in the structure of the noise for a wide class of distribution functions; that is, an adaptive procedure is proposed and investigated by means of which the threshold is adjusted to maintain an invariant false-alarm rate when the noise medians are rapidly changing and/or unknown. Conditions unddr which the adaptive T-detector remains distribution-free are also obtained. The adaptive T-detector is then applied to the detection of a gaussian signal in gaussian noise and its performance in the problem evaluated and compared to the performances of the T-detector and optimum likelihood detector.

1.4 Preview of Subsequent Chapters

In Chapter 2, the detection criterion utilized in this investigation and the assumptions on which it is based are discussed. The realizability of the assumptions is shown. In the same chapter, suitable means for comparing the distribution-free and likelihood detectors are proposed and their physical significance discussed.

In Chapter 3, coincidence detection procedures with invariant false-alarm rates for a wide class of distribution functions are proposed. In particular, the median detector, based on a specific distribution-free coincidence procedure, is investigated and its general properties obtained. Subsequently, the median detector is applied to various detection problems of practical importance; its performance is evaluated and compared to the performance of likelihood detectors.

In Chapter 4, the median detector is made a learning system with respect to unknown stationary or quasi-stationary medians. Conditions under which the learning median detector false-alarm rate remains distribution-free are also obtained. Subsequently, the learning median detector is applied to wo detection problems of practical importance and its performance and learning efficiency are evaluated and compared to the performance and learning efficiency of likelihood detectors.

In Chapter 5, the median detector is made adaptive to non-stationary noise medians. Conditions under which the adaptive median detector false-alarm rate remains destribution-free are obtained. The adaptive median detector is then applied to various detection problems and its performance is evaluated and compared to that of likelihood detectors.

In Chapter 6, detection based on the T-statistic is investigated and its general properties examined. Conditions under which the T-detector false-alarm rate remains distribution-free are also obtained. The T-detector is then applied to the problem of detecting a gaussian signal in gaussian noise, and its performance in the problem is evaluated and compared to that of a likelihood detector.

In Chapter 7, the T-detector is made adaptive with respect to rapidly changing and/or unknown data and reference channel noise medians. The conditions under which the adaptive T-detector false-alarm rate remains distribution-free are obtained. The adaptive T-detector is then applied to the detection of a gaussian signal in gaussian noise and its performance is evaluated and compared to that of a likelihood detector.

In Chapter 8, conclusions are drawn and areas for future work are indicated.

Chapter 2

GENERAL CONSIDERATIONS

2.1 Introduction

In this chapter, the detection criterion utilized in this investigation and the assumptions on which it is based are presented and discussed. It is shown that the assumptions are reasonable and the conditions implied are realizable, under certain conditions.

In the weak signal case, the test statistics employed here obey a set of regularity conditions. These are stated and their significance is discussed. A performance relation which has been previously derived from the regularity conditions is also presented. For a given detection problem, it relates the information rate, signal-to-noise ratio, and the error probability to a constant which is characteristic of the detector used. Thus, the above constant may be utilized as an index of performance of the detection system.

For comparing the distribution-free detectors to their equivalent likelihood detectors, suitable means are proposed and their physical significance discussed. These are the asymptotic relative efficiency and the detector output signal-to-noise ratio. The asymptotic relative efficiency is shown to be a measure of the relative information rate for a specified error probability and vanishing input signal-to-noise ratios. The output signal-to-noise ratio is defined to be the difference between the means of the test statistic under no-signal conditions divided by the variance of the statistic under signal conditions. It is shown, for the weak signal case treated here, that the output signal-to-noise ratio as defined above is functionally related to the error probability. The exact functional relation is also given.

2.2 The Detection Criterion

The detection criterion utilized in this investigation is based on the following assumptions. It is assumed that:

a. it is possible to obtain a sample function $N(t)$ of the noise random process $\{N(t)\}$, $N(t)$ hereon to be referred to as the reference sample function;

b. it is possible to obtain a sample function $Y(t)$ of the channel output stochastic process $\{Y(t)\}$, $Y(t)$ hereon to be referred to as the data sample;

c. it is possible to obtain n independent samples $Y_1, Y_2, \ldots Y_n$ from the sample function $Y(t)$ and m independent samples $X_1, X_2, \ldots X_m$ from the sample function $N(t)$;

d. in the absence of the signal, $\{Y(t)\}$ and $\{N(t)\}$ are two stochastic processes of identical first order distributions

On the basis of the samples $Y_1, Y_2, \ldots Y_n$ and $X_1, X_2, \ldots X_m$, a decision procedure for detecting deterministic or stochastic signals in noise is formulated by testing:

H_o: probability distribution function (cdf) of Y_i is $G_o(y)$, $i = 1,2, \ldots n$ and cdf of X_i is $F_o(x)$, $i = 1,2, \ldots m$ and such that $F_o(y) = G_o(y)$; signal is absent

against

H_1: probability distribution function of Y_i is $G_{\theta_i}(y)$ and that of X_i is $F_o(y)$ and such that $G_{\theta_i}(y) \neq F_o(y)$; signal is present where $F_o(y)$ is the distribution function of X_i when the signal is present or absent, $G_o(y)$ is the distribution function of Y_i when the signal is absent, and $G_{\theta_i}(y)$ is the distribution of Y_i in the presence of signal. It is to be noted that $G_{\theta_i}(y)$ depends both on the index i and θ, the signal-to-noise ratio parameter.

The above decision procedure simply states that if the signal is absent, then the distribution of Y_i must be the same as the distribution of X_i since both were obtained from stochastic processes of identical first order statistics under no-signal conditions. If, however, the signal is present, then the distribution function of Y_i is not the same as that of X_i since the samples Y_i were obtained from a sample function of the signal and noise process $\{Y(t)\}$ while the samples X_i were obtained from a sample function of the noise only random process $\{N(t)\}$.

From the previous discussion of the detection criterion and its associated assumptions, it is obvious that the acquisition of the reference sample function from the noise random process and the extraction of independent samples from the data and reference sample functions are matters of cardinal importance.

The acquisition of the reference sample function may be accomplished in various ways depending on the nature of the noise process and the requirements on information rate. If the noise process is stationary, then N(t) can be obtained once and for all before the transmission of information commences. From N(t) the m samples will then be obtained and stored in the receiver, to be compared later with the n data samples obtained from Y(t). If the noise random process is quasi-stationary — that is, if the noise statistics although varying with time do so rather slowly in comparison with the signaling rate — then one obtains the m noise samples from the noise entering the receiver when it is known that only noise exists in the channel, and uses them only for as long as the noise process remains stationary. Whenever the noise statistics begin to change, the transmission of information must be interrupted for a sufficient time to enable the receiver to collect a new set of m samples to be used for as long as the noise remains stationary. The previous procedure used for acquiring the m noise samples when the noise statististics are quasi-stationary requires knowledge of the length of the time interval for which the noise statistics are stationary. Such knowledge may be had as a result of experimental or theoretical investigations. Another disadvantage of the above procedure is that it requires interrupting the transmission of information, with a consequent reduction in information rate. If such reduction of information rate is undesirable, or if knowledge concerning the length of time during which the noise statistics remain stationary is not forthcoming, one may employ space, angle or frequency diversity to secure a channel containing noise only. In selecting the channels, care must be taken to insure that the first-order noise statistics will be the same in the reference and data channels. However, even after a careful selection of the reference channel, it is still possible that differences between the first order probability distributions of the reference channel and data channel noise random processes will exist. To guard against erroneous decisions resulting from such differences, two of the test statistic treated here are made adaptive to differences in noise statistics or to quasi-stationary variations in the noise statistics.

If the channel statistics are non-stationary, that is, if the variations in statistics occur at a rate comparable to the signaling rate, it is evident that one must by necessity employ diversity in order to secure a reference channel and a data channel such that the noise processes present in the channels have identical first order statistics

The assumption of independent samples is of prime importance and at the basis of every result obtained concerning distribution-free statistics. However, despite the importance attached to it, no sampling procedure has been proposed to date that would enable one to obtain independent samples without knowledge of the statistics of the process. The statement is usually made that to obtain independent samples one must sample infrequently. However, no quantitative measure has been given of the length of time between samples required to insure the independence of the samples. Admittedly, the subject is a very difficult one. A promising approach to the problem, at least for stationary or sufficiently quasi-stationary random processes, may be found in distribution-free tests of independence. Distribution-free tests of independence have been studied extensively in the statistical literature (18,19). The tests require n pairs (x, y) of samples from a continuous distribution function $F(x, y)$ with continuous marginal distribution functions $G(x)$, $H(y)$. They are used to test the hypothesis H_0: $F(x, y) = G(x)\, H(y)$, for all x, y. In applying the tests to the communication problem, one would obtain the sample pairs from the channel output sample function as shown in figure 1. The time τ is the time allowed between samples and the question to be answered is whether it is sufficiently long to insure the independence of the samples. It is seen, from figure 1, that to collect n pairs of samples (x, y) where x and y are τ seconds apart requires considerable time. During this time the first and second order statistics of the process must remain the same in order for the test to be applicable — hence the necessity for stationarity or quasi-stationarity of the random processes.

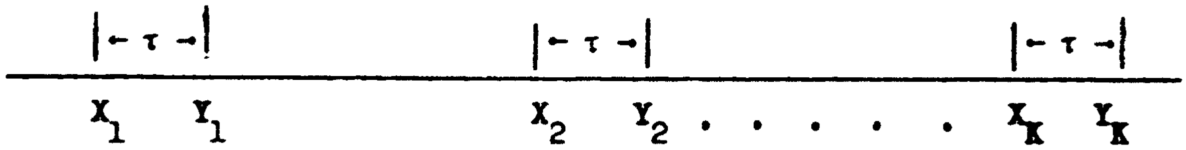

Fig. 1. Sampling for Test of Independence

2.3 <u>Means of Comparison</u>

A detection theory to be complete must a) suggest the structure of the detection system, b) specify procedures for evaluating the performance of a particular system, and c) specify means for comparing various systems. A choice of one of the many distribution-free test statistics specifies the detector structure. Thus, distribution-free detection theory fulfills the first of the above requirements. In the following, means for evaluating the performance of a detector and means for comparing it to the performance of other detectors will be given. To facilitate the presentation, a set of regularity conditions that the detectors investigated here obey will be stated and their significance discussed. The conditions are:

(A) $\dfrac{S_n - E_\theta(S_n)}{\sigma_\theta(S_n)}$ is asymptotically gaussian with mean zero and variance one uniformly for θ in the closed interval $[o, a]$, $a > o$, $\sigma_\theta(S_n) > o$, where S_n is a distribution-free statistic and $E_\theta(S_n)$, $\sigma_\theta^2(S_n)$ are its mean and variance, respectively.

(B) $E'_\theta(S_n) = \dfrac{d}{d\theta} E_\theta(S_n)$ exists for all θ in (o, a), and is continuous at $\theta = 0$.

(C) $\lim\limits_{n \to \infty} \dfrac{1}{n}\left[\dfrac{E'_\theta(S_n)}{\sigma_\theta(S_n)}\right]^2_{\theta=0} = K$ where K is a positive constant;

(D) there exists a sequence $\{\theta_n\}$ such that $\lim\limits_{n \to \infty} \theta_n = 0$

$$\text{(E)}\ \lim_{n \to \infty} \frac{\sigma_\theta(S_n)}{\sigma_0(S_n)} = 1$$

$$\text{(F)}\ \lim_{n \to \infty} \sigma_0^{\,2}(S_n) = 0$$

Condition (A) simply states that the test statistic S_n is asymptotically gaussian both under no-signal conditions ($\theta = 0$) and under signal conditions ($\theta \neq 0$). Thus, accoring to condition (A), the general character of the test statistic S_n, obtained when m and n are large, is shown in Fig. 2.

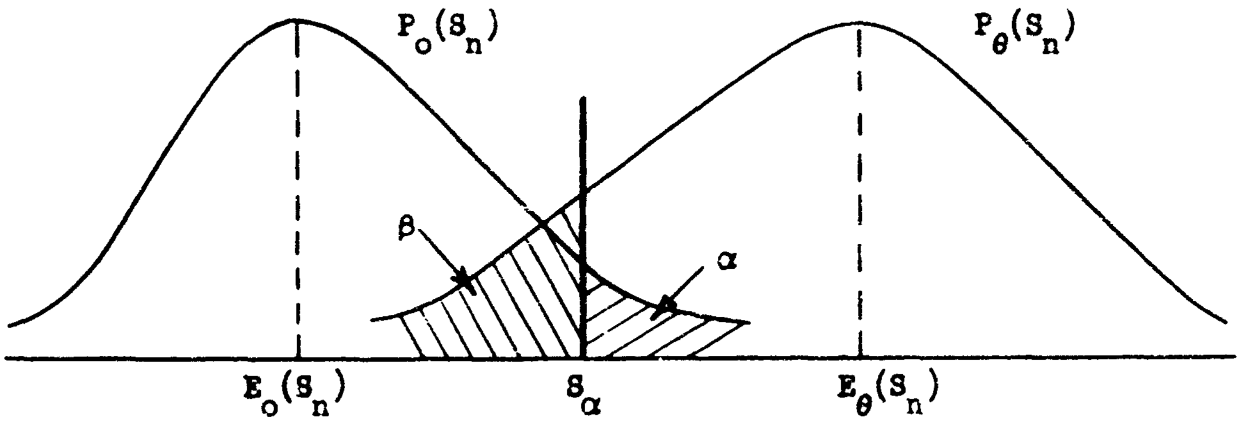

Fig. 2. Test Statistic Distributions for Large Sample Sizes

Here S_α is a decision threshold chosen to insure a probability of false alarm α. The parameter β is the false dismissal probability and $1 - \beta$ is the detection probability. Conditions (B), (C), (E) and (F) are self-explanatory. In connection with condition (C), it should be noted that K is independent of the number of samples and the input signal-to-noise ratio parameter θ_n. It depends only on a functional of the noise and signal and noise only distributions and the particular test statistic utilized. Condition (D) simply states that we are considering a sequence of alternatives which approach the hypothesis H_o of no signal present as the number of samples increases. The sequence of alternatives specified by condition (D) is necessitated by a desire to maintain constant detection probability for a constant false-alarm probability as the number of samples increases. The truth of the above statement will become obvious by examining Figure 2, and applying conditions (E) and (F). The desire to maintain constant false-alarm and detection probabilities as the number of samples increases stems from the fact that under constant false-alarm and detection probabilities as n increases, an explicit functional relation exists for comparing the information rates of two detectors under identical conditions. This will become apparent when the concepts of asymptotic relative efficiency and output signal-to-noise ratio are discussed. A practical consequence of condition (D) is that any result obtained based on the above conditions is valid for vanishing signal-to-noise ratios.

The results obtained in this investigation were derived on the basis of the aforementioned regularity conditions. Thus, a restriction of the level of generality was made by considering only the detection of weak signals in noise. This is appropriate since the weak signal case is usually the least amenable to solution and the case one usually desires to solve in practice. Moreover, as was pointed out previously, the choice of weak signals will also make possible explicit functional expressions for the means of comparison.

It has been shown (16) that a detector based on a test statistic that satisfies the regularity conditions possesses for large sample sizes a performance relation given

$$\lim_{n \to \infty} K\theta_n^2 n = 2\left[\operatorname{erf}^{-1}(1 - 2\alpha) + \operatorname{erf}^{-1}(1 - 2\beta)\right]^2 \qquad (2.3\text{-}1)$$

This relates the probability of error, the input signal-to-noise ratio θ_n, and the number of samples n to K. The constant K is dependent on the test statistic and the statistics of the detection problem under consideration. The importance of the parameter K is apparent. It has been called by Pitman (20) the efficacy of the test statistic, and it may be utilized as an index of performance for the detector using the statistic. It will be seen subsequently that both the asymptotic relative efficiency and the output signal-to-noise ratio are functionals of the efficacy.

2.3.1 Asymptotic Relative Efficiency

One of the most important considerations in a detection problem is the length of time required to detect the presence or absence of the signal with specified accuracy α and β, since the signaling rate and hence the information rate depend on the detection time. The only time consumed by a distribution-free detector utilizing a data and a reference channel is the time required to obtain the n samples from the data channel. If the condition of independence is imposed on the samples, then there is a limit on how closely one may sample and still obtain independent samples. Hence, the number of samples required for detection is inversely proportional to the information rate.

A detector based on the test statistic S_n can be compared to the detector based on the test statistic L_n on the basis of the information rate R possible with S_n vs. the information rate R^* possible with L_n, for the same signal in the same environment and for the same specified probability of error. The comparison will be based on the asymptotic relative efficiency (20,21,22) defined as:

$$\begin{aligned} ARE_{S_n,L_n} &= \frac{R}{R^*} \\ &= \frac{n^*}{n} \end{aligned} \qquad (2.3\text{-}2)$$

where n^* are the samples required by L_n and n are the samples required by S_n for the same probability of error.

If the test statistics satisfy conditions (A)-(F), then utilizing the performance relation (2.3-1), we obtain:

$$ARE_{S_n,L_n} = \frac{K}{K^*} \qquad (2.3\text{-}3)$$

where K and K^* are the efficacies of S_n and L_n, respectively, defined by condition (C). Thus, the asymptotic relative efficiency is in this case a measure of how much better is the information rate of the detector based on the statistic S_n than the information rate of the detector based on the statistic L_n in the detection of the same weak signal in the same environment with a specified error probability.

2.3-2 Signal-to-Noise Ratio

The output signal-to-noise ratio of a detector based on the test statistic S_n is defined to be:

$$\left(\frac{S}{N} \right) = \frac{E_\theta(S_n) - E_o(S_n)}{\sigma_\theta(S_n)} \qquad (2.3\text{-}4)$$

As a consequence of condition (B), the mean value theorem (23) may be applied to obtain

$$E_\theta(S_n) - E_o(S_n) = \theta_n \left[E'_\theta(S_n) \Big|_{\theta = \hat{\theta}} \right]$$

where $0 < \theta < \theta_n$. Utilizing this result in Eq. (2.3-4), we have

$$\underset{n \to \infty}{\text{limit}} \left(\frac{S}{N} \right) = \frac{\theta \left[E'_\theta(S_n) \Big|_{\theta=0} \right]}{\sigma_\theta(S_n)} \tag{2.3-5}$$

which, as a consequence of the continuity of $E'_\theta(S_n)$ at $\theta = 0$, and conditions (C), (D) and (E), can be rewritten as

$$\underset{n \to \infty}{\text{limit}} \left(\frac{S}{N} \right) = \theta \sqrt{nk} \tag{2.3-6}$$

Thus, it is seen from Eq. (2.3-6) that the output signal-to-noise ratio for large sample sizes and vanishing signal-to-noise ratios is proportional to the input signal-to-noise ratio. The constant of proportionality is a function of the sample size n and the efficacy K. Again the importance of the detector efficacy is evident.

The physical justification for the concept of output signal-to-noise ratio, as defined above, becomes apparent if one studies Fig. 2. It is observed there that the two patterns, signal absent and signal present, become more distinguishable as either the distance between their central locations (mean values) becomes greater or the concentration of their values around the central locations becomes greater (smaller variances). It is also apparent that any increase in distinguishability between the patterns due to increased distance between their central locations will be nullified by an increase in variance. The same is true for a decrease in variance if accompanied by a decrease in distance. Hence, it is the ratio of distance between central locations of the two patterns to the variance of the signal present pattern that can serve as a measure of the distinguishability of the patterns or detectability of the signal in noise, in the sense described above. In fact, the output signal-to-noise ratio, as defined here, is a measure in a given detection problem of the false dismissal probability for a specified false alarm probability.

Thus, the output signal-to-noise ratio serves as a qualitative measure of the detectability of the signal in a given environment utilizing a specified detector. In particular, for weak signals and large number of samples, an exact expression of the relation between the probabilities of false alarm and false dismissal and the output signal-to-noise ratio may be derived utilizing Eq. (2.3-6) and the performance relation Eq. (2.3-1). The functional relation between output signal-to-noise ratio and false alarm and false dismissal probabilities is

$$\left(\frac{S}{N} \right) = 2 \left[\operatorname{erf}^{-1} (1 - 2\alpha) + \operatorname{erf}^{-1} (1 - 2\beta) \right]^2 \qquad (2.3\text{-}7)$$

Chapter 3

MEDIAN DETECTOR

Introduction

Coincidence detection procedures base their decisions on the presence or absence of the signal on the following test statistic

$$S_n = \frac{1}{n} \sum_{i=1}^{n} c(y_i - x) \tag{3.1-1}$$

where y_i are observations on the input waveform $Y(t)$, and x is the threshold level. The function $c(z)$ is defined as

$$\begin{aligned} c(z) &= 1, \qquad z > 0 \\ &= 0, \qquad z \leq 0 \end{aligned} \tag{3.1-2}$$

The mean and variance of S_n under no-signal conditions are

$$\begin{aligned} E_o[S_n] &= \frac{1}{n} \sum_{i=1}^{n} E_o[c(Y_i - x)] \\ &= \frac{1}{n} \sum_{i=1}^{n} \int_x^{\infty} d\, F_{o_i}(y) \\ &= \frac{1}{n} \sum_{i=1}^{n} [1 - F_{o_i}(x)] \\ &= 1 - \frac{1}{n} \sum_{i=1}^{n} F_{o_i}(x) \end{aligned} \tag{3.1-3}$$

and

$$\sigma_o^2[S_n] = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{\theta_i}^2 [c(Y_i - x)] \qquad (3.1\text{-}4)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} [1 - F_{o_i}(x)][F_{o_i}(x)]$$

where $F_{o.}(y)$ is the probability distribution function of the random variable Y_i

The test statistic S_n is equal to a sum of independent, binomially distributed random variables; hence, it follows from the central limit theorem (22) that S_n is asymptotically gaussian under signal and under no-signal conditions.

For large number of samples, the distribution of S_n, being gaussian, is completely specified by its mean and variance. In turn, the false alarm rate of S_n is completely specified by the distribution of the test statistic under no-signal conditions. Thus, for an invariant or distribution-free false alarm rate, the mean and variance of S_n under no-signal conditions must be constant, and independent of the channel statistics. It is seen from Eqs. (3.1-3) and (3.1-4) that if the threshold level x is chosen as in previous investigations (14,15,16,17), the mean and variance of the test statistic S_n will vary according to the distribution function of the noise under no-signal conditions. Thus, the false alarm rate of the coincidence detection procedure will also vary. For the reasons stated in Chapter 1, the latter is undesirable when the channel statistics are unknown. A need, therefore, exists for coincidence detection procedures with invariant or distribution-free false alarm rates.

A distribution-free coincidence procedure results if the threshold level is chosen to be a specified noise quantile under no-signal conditions. Thus, if the pth quantile z_p is chosen as the threshold level, the mean and variance under no-signal conditions become

$$E_o[S_n] = 1 - \frac{1}{n} \sum_{i=1}^{n} F_{o_i}(z_p) \tag{3.1-5}$$

$$\sigma_o^2[S_n] = \frac{1}{n^2} \sum_{i=1}^{n} [1 - F_{o_i}(z_p)][F_{o_i}(z_p)] \tag{3.1-6}$$

For noise with stationary quantile z_p under no-signal conditions, we have

$$F_{o_i}(z_p) = p, \qquad i = 1, 2, \ldots, n \tag{3.1-7}$$

and the mean and variance are known constants given by

$$E_o[S_n] = 1 - p \tag{3.1-8}$$

$$\sigma_o^2[S_n] = \frac{p(1 - p)}{n} \tag{3.1-9}$$

Thus, for this choice of threshold level, the distribution of the coincidence detection procedure test statistic S_n is asymptotically known and independent of the channel statistics. Hence, the coincidence detection procedure possesses an invariant or distribution-free false alarm rate.

To utilize the distribution-free coincidence procedures proposed here, the only information needed concerning the channel statistics is the specified quantile of the noise under no-signal conditions.

In this chapter, a particular distribution-free coincidence detector is proposed and investigated in detail. It utilizes the median of the noise under no-signal conditions as its threshold level — hence the name median detector for this coincidence detection procedure. In essence, the median detector tests for the presence of the signal by testing for a change of the median of the input waveform $Y(t)$, the assumption being made implicitly that such a change in median is the result of the presence of the signal.

In the following, the general properties of the median detector test statistic are obtained. In particular, its efficacy, output signal-to-noise ratio, and performance relation are obtained. Subsequently, the median detector is applied to various detection problems of interest and its performance is evaluated and compared to that of likelihood detectors.

3.2 The Median Detector Test Statistic

The median detector as defined above is based on the following test statistic

$$S_n(M) = \frac{1}{n} \sum_{i=1}^{n} c(y_i - M) \tag{3.2-1}$$

where the threshold level M is the median of the noise under no-signal conditions, and the function $c(z)$ is defined in Eq. (3.1-2). Therefore, the test statistic is operating on the input waveform $Y(t)$ in the same manner as the system shown in Fig. 3.

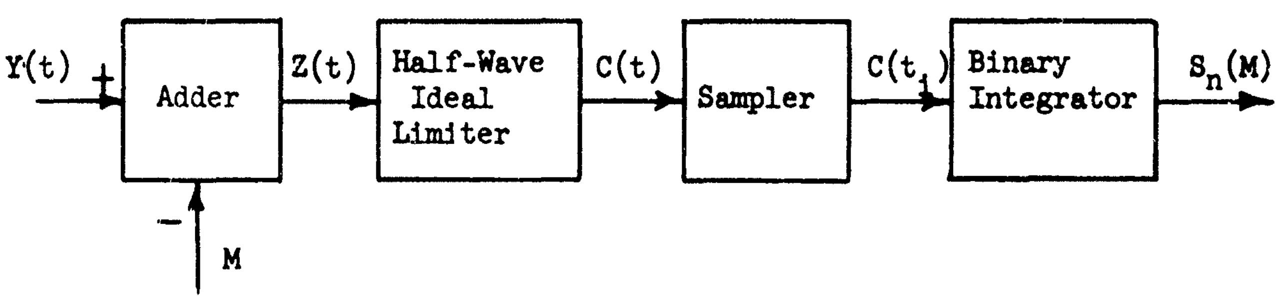

Fig. 3. Block Diagram of Median Detector

The median M is subtracted from the data sample function $Y(t)$; the resulting waveform $Z(t)$ is then applied to an ideal half-wave limiter the output of which is sampled n times, and the samples averaged to yield $S_n(M)$.

The test statistic $S_n(M)$ decides for the presence of the signal by testing for a difference between the median of the data sample function and M, the median of the reference or noise only sample function. In effect, the median M serves as the reference sample function.

Stated explicitly, the conditions on which the operation of the median detector is based are

(a) the medians of the reference and data sample functions under no-signal conditions are the same;

(b) the common median M of the reference and data samples under no-signal conditions is known.

Condition (a) insures that any difference between the reference and data sample function medians is brought about by the presence of the signal. Condition (b) permits the calculation of the statistic $S_n(M)$. Both conditions will be met if the noise under no-signal conditions has a stationary median and the reference and data samples are obtained from the same channel. The stationarity of the noise median guarantees that the reference and data samples obtained from the same channel, hence from the same stochastic process $\{N(t)\}$ under no-signal conditions, will have the same median. The particular value of the median can be obtained by taking measurements for sufficiently long time on the channel before the transmission of information commences, so that the true value of the median is accurately known. This value of the median may be used for all time thereafter since the noise median is stationary.

The disturbance present in every communication channel consists (28) of an additive disturbance and a multiplicative one, the latter present only under signal conditions. Thus, the mild restrictions imposed on the noise under no signal conditions are restrictions on the additive noise only. No restrictions, whatsoever, on the nature of the multiplicative disturbance are necessary for the median detector to possess a distribution-free false alarm rate.

3.3 Median Detector General Properties

The general properties of the median detector will be obtained for the case of channel statistics that are first order stationary under no-signal conditions. Thus, as a consequence of the first order stationarity, the random variables $Y_1, Y_2, \ldots, Y_n$, representing the amplitude of the data waveform at the sampling instants $t_1, t_2, \ldots, t_n$, are identically distributed with a common distribution $F_o(y)$, under no-signal conditions. Under signal conditions and in the presence of multiplicative disturbance, the continuous parameter stochastic process $\{Y(t)\}$ is not stationary since the signal strength at the receiver is varying with time. Thus, the distribution function of Y_i, $i = 1, 2, \ldots, n$ is not the same as that of Y_j, $j = 1, 2, \ldots, n$, $j \neq i$. However, we shall assume that the distribution of Y_i, $i = 1, 2, \ldots, n$ differs from the distribution of Y_j, $j = 1, 2, \ldots, n$, $j \neq i$, only through the signal-to-noise ratio parameter θ; i.e., the distribution of Y_i is $G_{\theta_i}(y)$ and that of Y_j is $G_{\theta_j}(y)$. This assumption is satisfied in many detection problems of interest.

The mean and variance of $S_n(M)$, under no-signal conditions, are given by Eqs. (3.1-8) and (3.1-9), where in this case

$$F_o(M) = \frac{1}{2} \tag{3.3-1}$$

Therefore the mean and variance are

$$E_o[S_n(M)] = \frac{1}{2} \tag{3.3-2}$$

$$\sigma_o^2[S_n(M)] = \frac{1}{4n} \tag{3.3-3}$$

It is seen that the mean and variance, under no-signal conditions, are known and constant, independent of the channel statistics. Thus, the median detector false alarm rate, as explained previously, is asymptotically distribution-free.

However, the mean and variance of $S_n(M)$ under signal conditions do depend on the channel statistics. Hence, the detection probability also depends on the channel statistics. The mean of $S_n(M)$ under signal conditions is given by

$$\begin{aligned} E_\theta[S_n(M)] &= \frac{1}{n}\sum_{i=1}^{n} E_{\theta_i}[c(\mathcal{Y}_i - M)] \qquad (3.3\text{-}4) \\ &= \frac{1}{n}\sum_{i=1}^{n} \int_M^\infty d\, G_{\theta_i}(y) \\ &= \frac{1}{n}\sum_{i=1}^{n} [1 - G_{\theta_i}(M)] \\ &= 1 - \frac{1}{n}\sum_{i=1}^{n} G_{\theta_i}(M) \end{aligned}$$

Applying the mean value theorem (23) we obtain

$$G_{\theta_i}(M)-G_o(M) = \theta_i \left[\frac{d\, G_\theta(M)}{d\theta} \bigg|_{\theta = \hat{\theta}} \right] \qquad (3.3\text{-}5)$$

where $0 < \hat{\theta} < \theta_i$, and because of the first order stationarity of the noise under no-signal conditions $G_o(M) = F_o(M)$. Substituting this result in Eq. (3.3-4), we obtain

$$E_\theta[S_n(M)] = 1 - F_o(M) - \frac{1}{n} \sum_{i=1}^{n} \theta_i \left[\frac{d\, G_\theta(M)}{d\theta} \bigg|_{\theta = \hat{\theta}} \right] \qquad (3.3\text{-}6)$$

$$= E_o[S_n(M)] - \frac{1}{n} \sum_{i=1}^{n} \theta_i \left[\frac{d\, G_\theta(M)}{d\theta} \bigg|_{\theta = \hat{\theta}} \right]$$

For the weak signal case, Eq. (3.3-6) becomes

$$E_\theta[S_n(M)] = E_o[S_n(M)] - \frac{1}{n} \sum_{i=1}^{n} \theta_i \left[\frac{d\, G_\theta(M)}{d\theta} \bigg|_{\theta = 0} \right] \qquad (3.3\text{-}7)$$

$$= E_o[S_n(M)] - \left[\frac{d\, G_\theta(M)}{d\theta} \bigg|_{\theta = 0} \right] \left[\frac{1}{n} \sum_{i=1}^{n} \theta_i \right]$$

$$= E_o[S_n(M)] - \bar{\theta} \left[\frac{d\, G_\theta(M)}{d\theta} \bigg|_{\theta = 0} \right]$$

where $\bar{\theta}$ is the average signal-to-noise ratio

$$\bar{\theta} = \frac{1}{n} \sum_{i=1}^{n} \theta_i \qquad (3.3\text{-}8)$$

The variance of the test statistic $S_n(M)$ under signal conditions is

$$\sigma_\theta^2[S_n(M)] = \frac{1}{n^2}\sum_{i=1}^{n} \sigma_{\theta_i}^2 [c(Y_i - M)] \tag{3.3-9}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} [1 - G_{\theta_i}(M)]\,[G_{\theta_i}(M)]$$

Utilizing Eq. (3.3-5), we obtain

$$\sigma_\theta^2[S_n(M)] = \sigma_o^2[S_n(M)] - \frac{1}{n^2}\sum_{i=1}^{n} \theta_i^2 \left[\left.\frac{d\, G_\theta(M)}{d\theta}\right|_{\theta = 0}\right]^2 \tag{3.3-10}$$

For the weak signal case, this becomes

$$\sigma_\theta^2[S_n(M)] = \sigma_o^2[S_n(M)] - \frac{1}{n^2}\,\overline{\theta^2}\left[\left.\frac{d\, G_\theta(M)}{d\theta}\right|_{\theta = 0}\right]^2 \tag{3.3-11}$$

where $\overline{\theta^2}$ is the mean-square value of the signal-to-noise ratio

$$\overline{\theta^2} = \frac{1}{n}\sum_{i=1}^{n} \theta_i^2 \tag{3.3-12}$$

It is to be noted that for the weak signal case $\overline{\theta^2} << \overline{\theta}$.

Utilizing Eqs. (3.3-3) and (3.3-7), we obtain the efficacy of $S_n(M)$

$$K[S_n(M)] = \lim_{n \to \infty} \frac{1}{n}\left[\left.\frac{E_\theta'[S_n(M)]}{\sigma_o[S_n(M)]}\right|_{\theta = 0}\right]^2 \tag{3.3-13}$$

$$= 4\left[\left.\frac{d\, G_\theta(M)}{d\theta}\right|_{\theta = 0}\right]^2$$

The test statistic $S_n(M)$, being the sum of independent binomially distributed random variables, is asymptotically gaussian under signal and under no-signal conditions. Thus, it satisfies condition (A).

Condition (D) is fulfilled in the weak signal case investigated here. The existence of the efficacy given in Eq. (3.3-13) is the only requirement for the statistic to satisfy conditions (B), (C) and (E). The efficacy exists for all continuous parameter stochastic processess with continuous first order distributions. It is seen from Eq. (3.3-3) that $S_n(M)$ satisfies condition (F) always.

A test statistic is said to be consistent if, for a specified false alarm rate, its detection probability approaches one as the number of samples increases. Conditions (E) and (F) establish the consistency of $S_n(M)$.

Since the test statistic satisfies conditions (A)-(F), its performance relation and output signal-to-noise ratio are given by

$$4\left[\left.\frac{d\,G_\theta(M)}{d\theta}\right|_{\theta=0}\right]^2 \bar{\theta}^2\, n = 2\left[\operatorname{erf}^{-1}(1-2\alpha) + \operatorname{erf}^{-1}(1-2\beta)\right]^2 \quad (3.3\text{-}14)$$

and

$$\left(\frac{S}{N}\right) = 2\,\bar{\theta}\,\sqrt{n}\left[\left.\frac{d\,G_\theta(M)}{d\theta}\right|_{\theta=0}\right] \quad (3.3\text{-}15)$$

The test statistic efficacy may also be used, as shown in Chapter 2, to obtain the relative information rate of $S_n(M)$ with respect to a likelihood statistic. Thus, the efficacy given in Eq. (3.3-13) completely specifies all of the performance indices of the median detector.

3.4 Applications

In the following, the median detector is applied to specific detection problems, its performance in the problem is evaluated and compared to that of likelihood detectors applicable to the problems.

3.4.1 Detection of a Sine Wave of Known Phase in Additive Noise - General Case

For this general problem, it has been shown, see Eq. (A-3) in Appendix A, that

$$\left.\frac{d\, G_\theta(y)}{d\theta}\right|_{\theta = 0} = - f_o(y) \qquad (3.4\text{-}1)$$

Thus

$$K[S_n(M)] = 4\, f_o^{\,2}(M) \qquad (3.4\text{-}2)$$

The efficacy of the likelihood detector appropriate for the problem is given by Eq. (B-25) in Appendix B as

$$K^* = 1 \qquad (3.4\text{-}3)$$

Therefore, the asymptotic relative efficiency is

$$ARE_{S_n(M),\, L^*_n} = 4\, f_o^{\,2}(M) \qquad (3.4\text{-}4)$$

It is seen from Eq. (3.4-4) above that the relative information rate of the median detector, with respect to the applicable likelihood detector for this general problem, may be anything from zero to infinity depending on the probability density $f_o(y)$ under no-signal conditions. However, Hodges and Lehman (29) have shown that for a probability density $f_o(x)$ which is non-increasing on either side of its median, the ARE is never less than 1/3, and this lower bound is attained when $f_o(x)$ is rectangular.

3.4.2 Detection of a Sine Wave of Known Phase in Additive Gaussian Noise

This problem is a specific case of the previous general detection problem, with $f_o(y)$ given by Eq. (A-4) as

$$f_o(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \qquad -\infty < y < \infty \tag{3.4-5}$$

thus, using Eq. (3.4-2) we obtain

$$K[S_n(M)] = 4\, f_o^{\,2}(M) \tag{3.4-6}$$

$$= \frac{2}{\pi}$$

The asymptotic relative efficiency is obtained, using Eq. (3.4-4), as

$$ARE_{S_n(M),\ L_n^*} = \frac{2}{\pi} \tag{3.4-7}$$

$$= 0.637$$

3.4.3 Detection of a Sine Wave of Known Phase in Additive Gaussian and Impulse Noise

This problem also is a specific case of the general problem. Hence, using Eq. (A-6) in Eqs. (3.4-2) and (3.4-4), we obtain

$$K[S_n(M)] = 4\, a^2 \tag{3.4-8}$$

$$= c^2 \frac{\Gamma\left(\frac{3}{c}\right)}{\Gamma^3\left(\frac{1}{c}\right)}$$

and

$$ARE_{S_n(M), L_n^*} = c^2 \frac{\Gamma\left(\frac{3}{c}\right)}{\Gamma^3\left(\frac{1}{c}\right)} \tag{3.4-9}$$

For $c = 2$, we obtain the asymptotic relative efficiency for gaussian noise only. This is, as was found previously, equal to 0.64. However, for $c = 1$, we obtain an asymptotic relative efficiency equal to 2.00; that is, for a purely exponential density characterizing the additive combination of impulse and gaussian noise, the median detector is twice as efficient as the likelihood detector designed under the gaussian assumption. The latter is a significant result. It points out that in the presence of additive gaussian and impulse noise — a combination found in many channels (11, 24) — it is advisable to utilize a median detector rather than a likelihood detector designed under the likelihood assumption.

3.4.4 Detection of a Sine Wave of Unknown Phase in Additive Gaussian Noise

In this problem we have from Eqs. (A-14) and (A-11) that

$$\left. \frac{d\, G_\theta(y)}{d\theta} \right|_{\theta = 0} = - \frac{y}{\sqrt{2\pi}\,\sigma^2}\, e^{-\frac{y^2}{2\sigma^2}}, \qquad -\infty < y < \infty \tag{3.4-10}$$

and $M = 0$ by symmetry of $f_o(y)$. Thus, using Eq. (3.3-13) we see that

$$\begin{aligned} K &= 4 \left. \frac{d\, G_\theta(M)}{d\theta} \right|_{\theta = 0} \\ &= 0 \end{aligned} \tag{3.4-11}$$

and consequently

$$ARE_{S_n(M),\, L_n^*} = 0 \tag{3.4-12}$$

In detecting a sine wave of unknown phase in additive gaussian noise, we may improve the efficiency of the distribution-free detector based on the test statistic $S_n(M)$ by proceeding in any of the following ways. We may discard the phase information by predetection processing the incoming waveform, e.g. envelope processing; or, we may change the specified quantile M to another specified quantile so that the detection efficiency increases and the detector still remains distribution-free.

If the distributions are known, then we may choose the threshold so that we maximize the ARE (16), thus obtaining an optimum coincidence procedure. However, it must be stressed that the last procedure requires complete knowledge of the first-order distributions. Thus, it does not apply to the detection problems with which this work is concerned.

3.4.5 Envelope Detection of a Sine Wave in Narrow-Band Gaussian Noise

Since the phase of the sine-wave is unknown, we may proceed to discard the phase information by envelope predetection processing the input waveform in an effort to improve the information rate of the detector. In this case, using Eq. (A-19) in Eq. (3.3-13), we obtain

$$K[S_n(M)] = 4\left[\frac{M^4}{4\sigma^4} e^{-\frac{M^2}{\sigma^4}}\right] \tag{3.4-13}$$

where M is given by

$$\frac{1}{2} = \int_0^M \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}} dy \tag{3.4-14}$$

thus

$$K[S_n(M)] = 0.48 \tag{3.4-15}$$

From Eq. (B-43) and Eq. (3.4-15) we obtain the asymptotic relative efficiency of the median detector with respect to the likelihood detector

$$ARE_{S_n}(M),\ L_n^* = 0.48 \tag{3.4-16}$$

Thus, the median detector is approximately half as efficient as the likelihood detector for this problem.

3.4.6 Square-Law Detection of a Sine Wave in Narrow-Band Additive Gaussian Noise

It is seen from Eq. (A-23) that

$$\left.\frac{d\,G_\theta(y)}{d\theta}\right|_{\theta=0} = -\,y\,e^{-y}\;, \qquad y > 0 \tag{3.4-17}$$

$$= 0 \;, \qquad y < 0$$

Substituting this in Eq. (3.3-13) we obtain the statistic efficacy

$$K[S_n(M)] = 4\,M^2\,e^{-2M^2} \tag{3.4-18}$$

where M the median under no-signal conditions is given by

$$\frac{1}{2} = \int_0^M e^{-y}\,dy \tag{3.4-19}$$

thus

$$K[S_n(M)] = 0.48 \tag{3.4-20}$$

Using Eq. (3.4-20) and Eq. (B-51) we obtain the asymptotic relative efficiency

$$ARE_{S_n}(M),\, L_n^* = 0.48 \tag{3.4-21}$$

It is the same as that obtained by envelope predetection processing the input waveform.

3.4.7 Envelope Detection of Narrow-Band White Gaussian Signal in Additive White Gaussian Noise

From Eq. (A-33) we obtain that

$$\left.\frac{d\, G_\theta(y)}{d\theta}\right|_{\theta = 0} = -\frac{y^2}{2\sigma_N^2}\, e^{-\frac{y^2}{2\sigma_N^2}} \quad , \quad y \geq 0 \tag{3.4-22}$$

$$= 0 \quad , \qquad\qquad y \leq 0$$

Substituting this in Eq. (3.3-13) we obtain the efficacy K

$$K = 4\, \frac{M^4}{4\sigma_N^2}\, e^{-\frac{M^2}{\sigma_N^2}} \tag{3.4-23}$$

where the median M under no-signal conditions is given by

$$\frac{1}{2} = \int_0^M \frac{y}{\sigma_N^2}\, e^{-\frac{y^2}{2\sigma_N^2}}\, dy \tag{3.4-24}$$

thus

$$K[S_n(M)] = 0.48 \tag{3.4-25}$$

Using Eq. (3.4-25) and Eq. (B-62) we obtain the asymptotic relative efficiency

$$ARE_{S_n}\ (M),\ L_n^* = 0.48 \tag{3.4-26}$$

Thus, the median detector is approximately half as efficient as the likelihood detector for this problem.

3.4.8 Square-Law Detection of Narrow-Band White Gaussian Signal in Additive White Gaussian Noise

Using Eqs. (A-35), (A-38) in Eq. (3.3-13) we obtain

$$K[S_n(M)] = 4\, \frac{M^2}{\sigma_N^4}\, e^{-\frac{2M}{\sigma_N^2}} \tag{3.4-27}$$

where the median M is given by

$$\frac{1}{2} = \int_0^M \frac{1}{\sigma_N^2} e^{-\frac{y}{\sigma_N^2}} dy \tag{3.4-28}$$

thus

$$K[S_n(M)] = 0.48 \tag{3.4-29}$$

Using Eq. (3.4-29) and Eq. (B-64) we obtain the asymptotic relative efficiency

$$\mathrm{ARE}_{S_n(M),\, L_n^*} = 0.48 \tag{3.4-30}$$

It is the same as that obtained by envelope predetection processing the input waveform.

3.5 Summary of Results

The median detector was found to possess many important properties. It was shown that the median detector is distribution-free in the sense that its false-alarm rate is constant, independent of the channel statistics as long as the median under no-signal conditions is known. It was also seen that the multiplicative disturbance does not affect the distribution-free nature of the median detector. Thus, as long as the non-stationarity of the channel statistics is confined to the statistics of the multiplicative disturbance, the distribution-free nature and the structure of the median detector are not affected. However, the detection probability of the detector does depend on the channel statistics.

In the case of the coherent detection of a sine wave in additive gaussian noise and no multiplicative disturbance, the information rate of

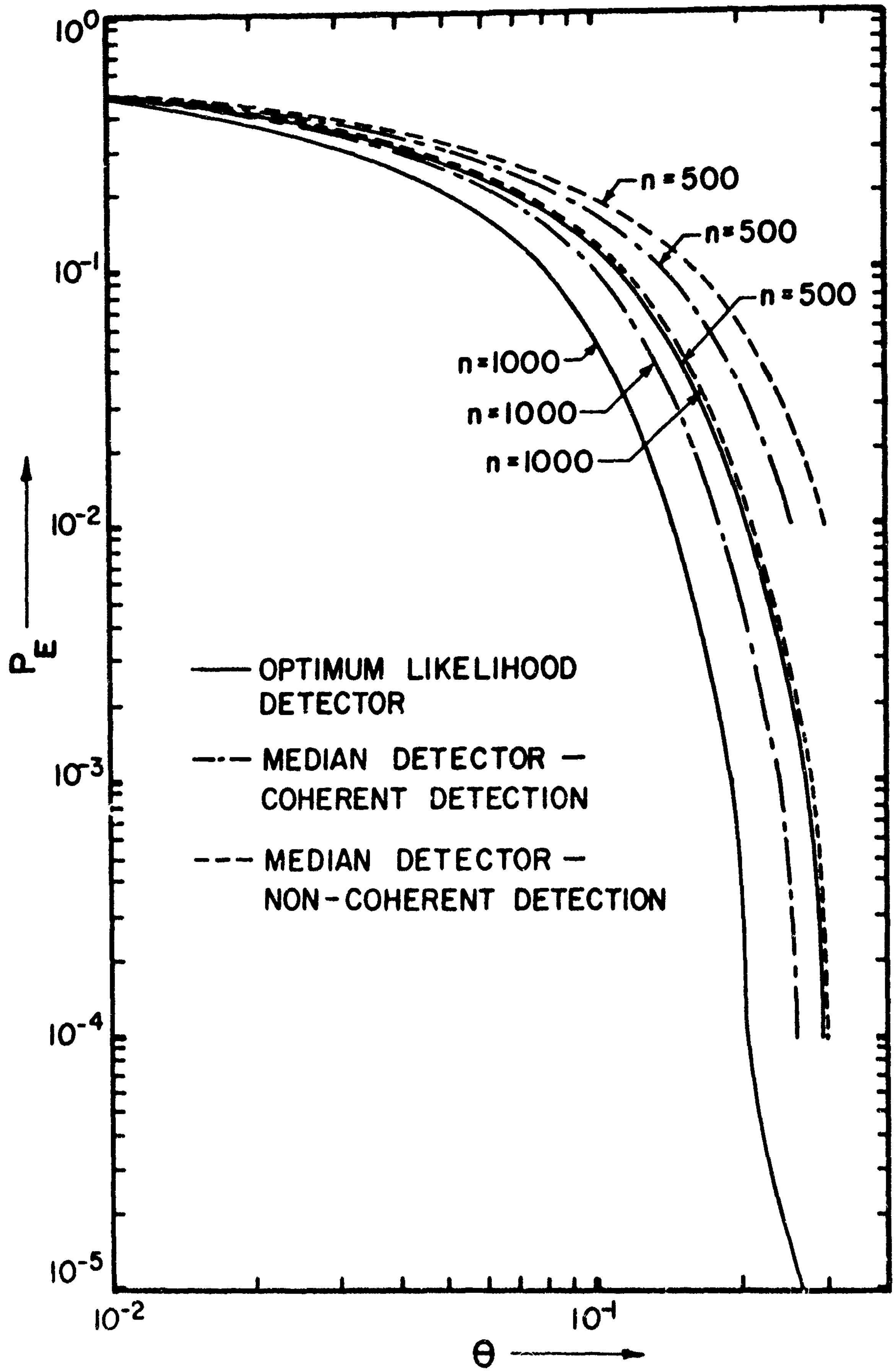

Fig. 4. Probability of Error vs. Received Mean SNR in the Detection of a Sine Wave in Gaussian Noise.

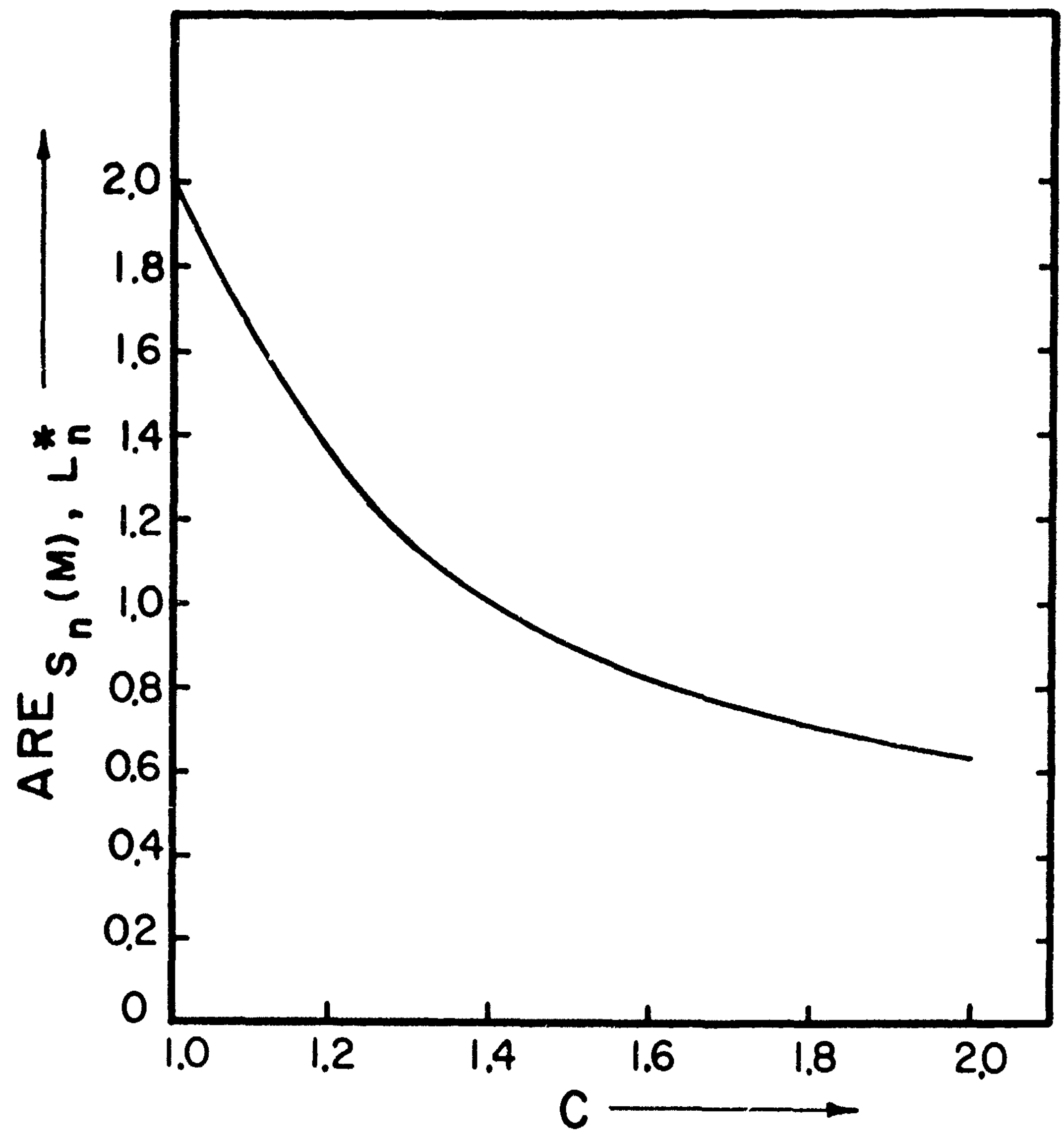

Fig. 5. Asymptotic Relative Efficiency vs. Parameter c in the Coherent Detection of a Sine Wave in Impulse and Gaussian Noise.

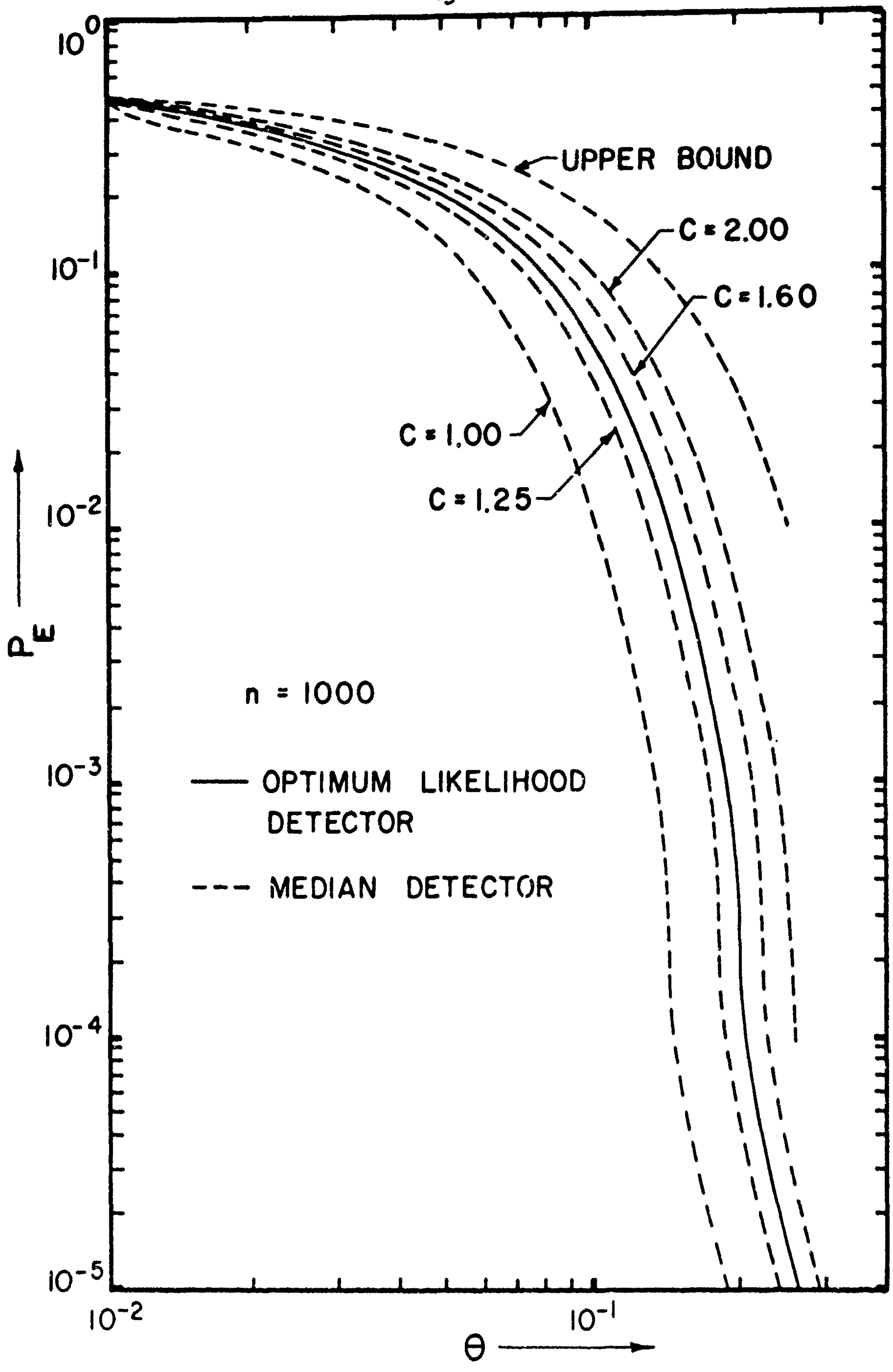

Fig. 6. Probability of Error vs. Received Mean SNR in the Coherent Detection of a Sine Wave in Additive Noise

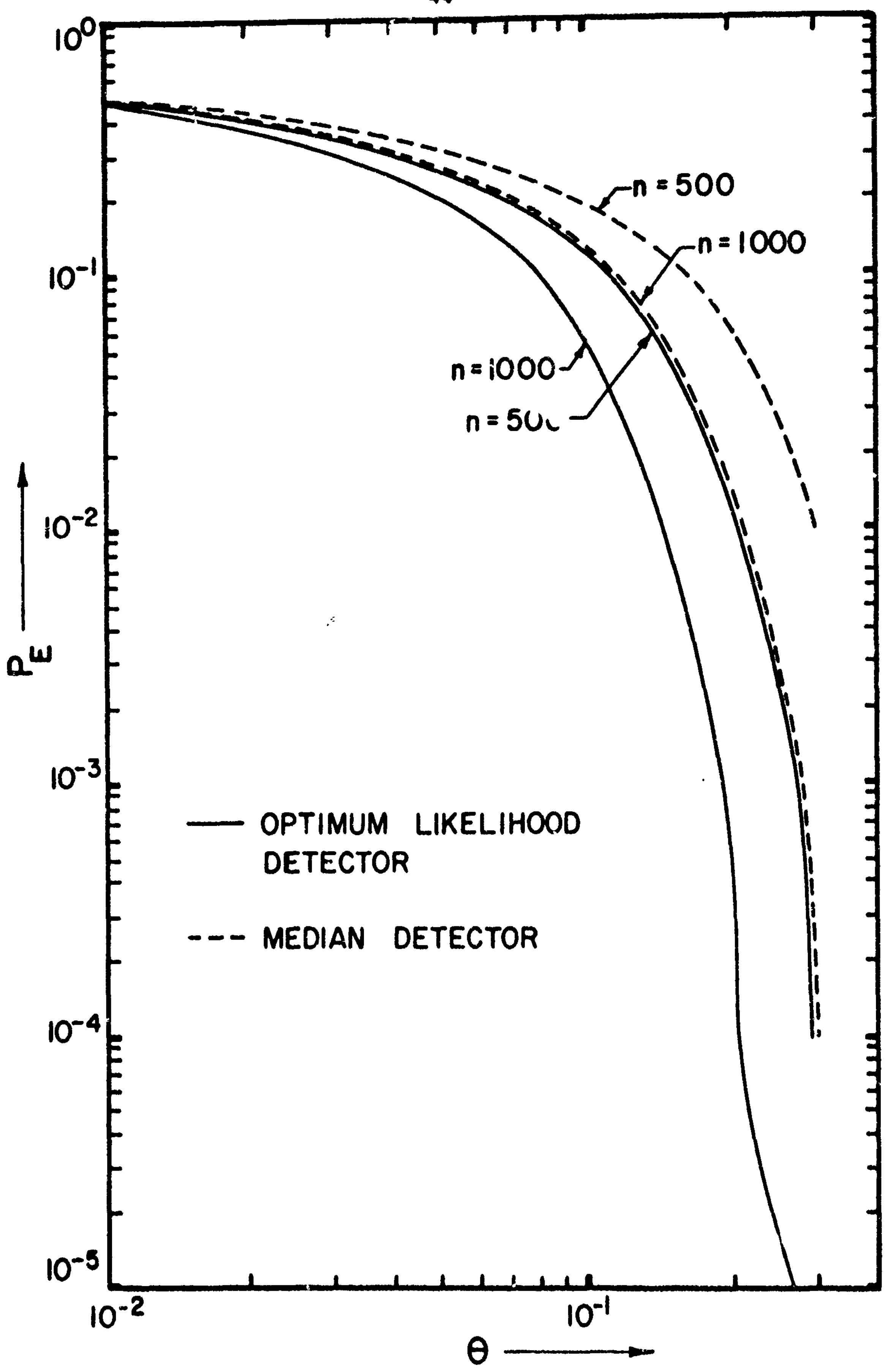

Fig. 7. Probability of Error vs. Received Mean SNR in the Detection of a Gaussian Signal in Gaussian Noise.

the median detector was found to be 64% of the information rate of the optimum likelihood detector. Or, in terms of the input signal-to-noise ratio, the optimum detector would require an input signal-to-noise ratio that is 80% of that required by the median detector for the same probability of error with the same number of samples. If the input waveform is predetection envelope or square-law processed, then the information rate of the median detector is 48% of the information rate of the optimum likelihood detector. Or, in terms of input signal-to-noise ratio, the likelihood detector would require an input signal-to-noise ratio that is 70% of that required by the median detector for the same probability of error and same number of samples.

A significant result appears when the channel statistics include additive disturbance — that is, an additive combination of gaussian and impulse noise. For this case, the information rate of the median detector may exceed that of the likelihood detector designed under the gaussian assumption and used in this problem. In fact, if the parameter c of the gaussian and impulse noise distribution is equal to one, the median detector information rate is twice that of the likelihood detector. Thus, the likelihood detector would require twice as many samples as the median detector to achieve the same error probability for the same input signal-to-noise ratio. Or, the median detector now would require a signal-to-noise ratio only 70% of that required by the likelihood detector for the same information rate and probability of error.

The results obtained in this chapter concerning the detection of a sine wave in gaussian noise are plotted in Fig. 4 while those concerning the detection of a sine wave in gaussian and impulse noise are shown in Fig. 6. The results pertaining to the detection of a stochastic signal in noise are plotted in Fig. 7. A graphical comparison of the performances, for various noise statistics, of the median detector in the detection of a sine wave of known phase in additive noise is given in Fig. 6.

From the results obtained in this chapter, it is concluded that the use of the median detector entails only a small loss in efficiency for guassian channel statistics; while in the presence of impulse noise, the use of the median detector may lead to higher efficiency depending on the distribution of the gaussian and impulse noise. In fact, the greater the impulse noise content, the higher the median detector efficiency. Moreover, the invariant structure in the phase of multiplicative disturbance, whether stationary or non-stationary, and the distribution-free nature of the false-alarm rate of the median detector add greatly to its appeal.

Chapter 4

LEARNING MEDIAN DETECTOR

4.1 Introduction

The median detector investigated in Chapter 3 tests for the presence of the signal by testing for a change in median under signal and under no-signal conditions. To do so, it utilizes a data sample which is compared with the median under no-signal conditions, the latter assumed to be known. However, knowledge of the median under no-signal conditions is not always forthcoming and the assumption of known median, in many practical cases, is not justified. For example, the median will be unknown when the detector is placed in an unknown environment and immediate operation of the detector is desired. In this situation, the detector must learn the unknown median while it is operating. The median detector will also have to go through a learning phase from time to time when the median is varying rather slowly with time. It is this learning phase of operation of the median detector that we are concerned with in this chapter.

In the present chapter, for stationary or at most quasi-stationary medians, the median detector test statistic is modified so that it learns the unknown median. To do so, the modified test statistic utilizes an estimator of the median. Conditions under which the modified test statistic remains distribution-free are obtained, and the learning nature of the detector based on the modified statistic is investigated. The learning detector is then applied to detection problems to which it is applicable, and its performance and learning efficiency are obtained and compared to those of the comparable likelihood detectors.

4.2 The Modified Test Statistic

The learning median detector is based on a test statistic that is a modified version of the median detector test statistic. The modified statistic is

$$S_n(\hat{M}) = \frac{1}{n} \sum_{i=1}^{n} c(y_i - \hat{M}_i) \qquad (4.2\text{-}1)$$

where $\hat{M}_i$ is an estimate, obtained from the reference waveform N(t), of the unknown median of the additive disturbance under no-signal conditions. The function c(z) was defined previously. The test statistic as defined above is operating on the input waveforms Y(t) and N(t) in the same manner as the system shown in Fig. 8.

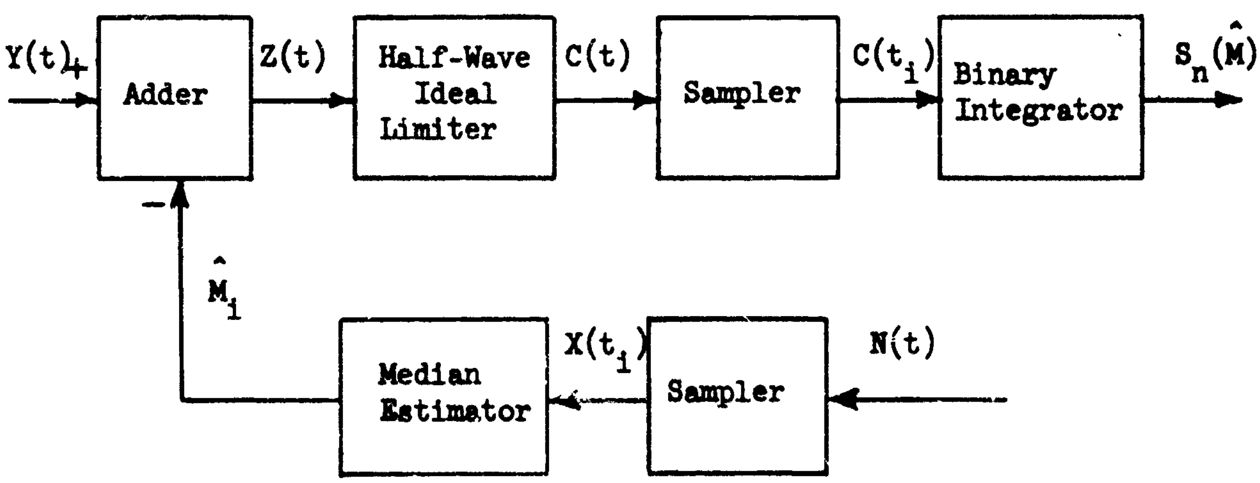

Fig. 8. Block Diagram of Learning Median Detector

The median estimate $\hat{M}_i$ obtained from N(t) is first subtracted from the data sample function Y(t), and the resulting waveform Z(t) is applied to an ideal half-wave limiter. The output of this limiter is then sampled n times, and the samples averaged to yield $S_n(\hat{M})$.

The test statistic $S_n(\hat{M})$ utilizes the reference sample function to estimate the unknown median under no-signal conditions, the assumption again being made that the medians of the reference and data samples are the same under no-signal conditions. In effect, the estimates of the median serve as the reference sample function for the modified test statistic. The conditions under which the medians will be the same were discussed in Chapter 3.

In utilizing the modified test statistic, the m reference samples are divided into n groups of $\frac{m}{n}$ samples each. From each of these n groups an estimate of the unknown median M is obtained. Each of these median estimates is associated with only one data sample. Thus, the median estimate $\hat{M}_i$ is associated with the data sample y_i. In this manner, for independent reference and data samples, $c(Y_i - \hat{M}_i)$ and $c(Y_j - \hat{M}_j)$ are independent random variables. The latter results in expressions for the mean and variance of $S_n(\hat{M})$ under no-signal conditions that are distribution-free for a wide class of distribution functions. Nevertheless, it must be pointed out that the estimating procedure proposed above is by no means the most efficient one.

4.2.1 Conditions for Distribution-free Modified Test Statistic

The modified test statistic false-alarm rate will be asymptotically distribution-free, provided the mean and variance of the test statistic are distribution-free under no-signal conditions. For channels with first order statistics that are stationary or quasi-stationary under no-signal conditions, the mean and variance under no-signal conditions are

$$E_o[S_n(\hat{M})] = \frac{1}{n}\sum_{i=1}^{n} E_o\,[c(Y_i - \hat{M}_i)] \tag{4.2-2}$$

$$= \frac{1}{n}\sum_{i=1}^{n} E_o\,[c(Z_i)]$$

$$= \frac{1}{n}\sum_{i=1}^{n} p\{Z_i > 0\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} [1 - F_z(0)]$$

$$= 1 - F_z(0)$$

and

$$\sigma_o^{\,2}[S_n(\hat{M})] = \frac{F_z(0)\,[1 - F_z(0)]}{n} \tag{4.2-3}$$

where $Z_i = Y_i - \hat{M}_i$, and $F_z(z)$ is the distribution function of Z under no-signal conditions. From the above expressions for the mean and variance, it is seen that a necessary and sufficient condition for the modified test statistic to be distribution-free is that zero be a specified quantile of the distribution of Z, regardless of the channel statistics. A sufficient condition for this to be true is given by the following theorem.

Theorem 4.1

For channels with symmetrical first order statistics under no-signal conditions, the random variable Z_i, $i = 1, 2, \ldots, n$, has a median of zero.

Proof:

The random variable Z_i was defined above as

$$Z_i = Y_i - \hat{M}_i \tag{4.2-4}$$

Thus, the probability density function of Z_i given by the integral

$$f_{Z_i}(z) = \int f_M(x)\, g_o(z + x)\, dx \tag{4.2-5}$$

where $g_o(y)$ and $f_M(x)$ are, respectively, the probability densities of Y_i and $\hat{M}_i$. For symmetrical first order channel statistics, $g_o(y)$ and the probability density $f_o(x)$ of X_i are symmetrical. It is well known (30) that for symmetrical densities, the mean and median coincide. Thus, in the case of symmetrical $f_o(x)$, the samp[illegible] may be chosen as the estimator of the unknown median. The probability density function of the sample mean for large number of samples is gaussian with mean equal to the unknown median M; that is, for reference and data samples with identical medians under no-signal conditions, both $f_M(x)$ and $g_o(y)$ are symmetrical about the median M. Thus, Y_i and $\hat{M}_i$, being symmetrical about the same point, have equal means; and Z_i, defined as the difference between the two, has a mean of zero. To prove that Z_i has also a zero mediar, it is only necessary to prove that the probability density of Z_i is symmetrical. The random variable may be expressed as

$$Z_i = Y_i - \hat{M}_i \tag{4.2-6}$$

$$= (Y_i - M) - (\hat{M}_i - M)$$

$$= U_i - V_i$$

and the probability density of Z_i is given by

$$f_{z_i}(z) = \int f_M(v)\, g_o(z+v)\, dv \tag{4.2-7}$$

where the densities $f_M(v)$ and $g_o(v)$ of the random variables V_i and U_i, respectively, are even functions. The density $f_{z_i}(z)$ may be written

$$f_{z_i}(z) = \int_{-\infty}^{0} f_M(v_i)\, g_o(z+v)\, dv + \int_{0}^{\infty} f_M(v)\, g_o(z+v)\, dv \tag{4.2-8}$$

and

$$f_{z_i}(-z) = \int_{-\infty}^{0} f_M(v)\, g_o(-z+v)\, dv + \int_{0}^{\infty} f_M(v)\, f_o(-z+v)\, dv \tag{4.2-9}$$

$$= \int_{0}^{\infty} f_M(-w)\, g_o(-z-w)\, dw + \int_{-\infty}^{0} f_M(-w)\, g_o(-z-w)\, dw$$

$$= \int_{0}^{\infty} f_M(w)\, g_o(z+w)\, dw + \int_{-\infty}^{0} f_M(w)\, g_o(z+w)\, dw$$

$$= f_{z_i}(z)$$

It is concluded from Eq. (4.2-9) that $f_{z_i}(z)$ is symmetrical about zero; thus, it has a median of zero. This completes the proof.

The above theorem establishes the distribution-free nature of the modified test statistic for a wide class of channels, namely the class with symmetrical first order statistics under no-signal conditions.

For symmetrical first order statistics, the mean and variance under no-signal conditions are

$$E_o[S_n(\hat{M})] = 1 - F_z(0) \qquad (4.2\text{-}10)$$
$$= \frac{1}{2}$$

and

$$\sigma_o^{\,2}[S_n(\hat{M})] = \frac{F_z(0)[1 - F_z(0)]}{n} \qquad (4.2\text{-}11)$$
$$= \frac{1}{4n}$$

Thus, the modified test statistic has asymptotically the same distribution under no-signal conditions as the median detector test statistic.

4.2.2 The Modified Test Statistic Efficacy

The mean and variance of $S_n(\hat{M})$ under signal conditions are

$$E_\theta[S_n(\hat{M})] = \frac{1}{n}\sum_{i=1}^{n} E_{\theta_i}\,[c(Y_i - \hat{M}_i)] \qquad (4.2\text{-}12)$$
$$= \frac{1}{n}\sum_{i=1}^{n} p(Y_i > \hat{M}_i)$$
$$= \frac{1}{n}\sum_{i=1}^{n} \int_{-\infty}^{\infty}\int_{x}^{\infty} d\,F_M(x)\,d\,G_{\theta_i}(y)$$
$$= \frac{1}{n}\sum_{i=1}^{n} \int_{-\infty}^{\infty}[1 - G_{\theta_i}(x)]\,d\,F_M(x)$$
$$= \frac{1}{n}\sum_{i=1}^{n} [1 - \int G_{\theta_i}(x)\,d\,F_M(x)]$$
$$= \frac{1}{n}\sum^{n} p(\theta_i)$$

and

$$\sigma_\theta^{\,2}[S_n(\hat{M})] = \frac{1}{n^2}\sum_{i=1}^{n} \sigma_{\theta_i}^{\,2}\,[c(Y_i - \hat{M}_i)] \tag{4.2-13}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} [\int G_{\theta_i}(x)\, dF_M(x)][1 - \int G_{\theta_i}(x)\, dF_M(x)]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} p(\theta_i)\,[1 - p(\theta_i)]$$

Applying the mean value theorem (23) to $G_{\theta_i}(y)$, we obtain in the weak signal case

$$G_{\theta_i}(y) - G_o(y) = \theta_i \left[\frac{d\,G_\theta(y)}{d\theta}\bigg|_{\theta=0}\right] \tag{4.2-14}$$

or, since $G_o(y) = F_o(y)$, we have

$$G_{\theta_i}(y) - F_o(y) = \theta_i \left[\frac{d\,G_\theta(y)}{d\theta}\bigg|_{\theta=0}\right] \tag{4.2-15}$$

Substituting this in Eqs. (4.2-12) and (4.2-13) we obtain

$$E_\theta[S_n(\hat{M})] = \frac{1}{n}\sum_{i=1}^{n}\left[1 - \int F_o(x)dF_M(x)\right] - \frac{1}{n}\sum_{i=1}^{n}\left[\int \frac{dG_\theta}{d\theta}\bigg|_{\theta=0} dF_M(x)\right]$$

$$= E_o[S_n(\hat{M})] - \bar{\theta}\left[\int \frac{dG_\theta(x)}{d\theta}\bigg|_{\theta=0} d\,F_M(x)\right] \tag{4.2-16}$$

and

$$\sigma_\theta^2[S_n(\hat{M})] = \frac{1}{n^2}\sum_{i=1}^{n}\int G_{\theta_i}(x)\, dF_M(x) - \frac{1}{n^2}\sum_{i=1}^{n}\left[\int G_{\theta_i}(x)\, dF_M(x)\right]^2$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left[\int F_o(x)\, dF_M(x)\right]\left[1 - \int F_o(x)\, dF_M(x)\right] \qquad (4.2\text{-}17)$$

$$- \frac{1}{n^2}\sum_{i=1}^{n}\theta_i^2\left[\int \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} d\, F_M(x)\right]^2$$

$$= \sigma_o^2[S_n(\hat{M})] - \frac{1}{n}\overline{\theta^2}\left[\int \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} d\, F_M(x)\right]^2$$

where $\bar{\theta}$ and $\overline{\theta^2}$ were defined in Chapter 3, respectively, as the mean and mean-square value of the signal-to-noise ratio θ.

Utilizing Eqs. (4.2-11) and (4.2-16), we obtain the efficacy of $S_n(\hat{M})$

$$K(\hat{M}) = 4\left[\int \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} d\, F_M(x)\right]^2 \qquad (4.2\text{-}18)$$

It will be shown in the next section that all of the performance indices of the detector utilizing the modified test statistic are completely specified by the modified test statistic efficacy. Thus, the learning nature of the detector may be examined by investigating the convergence properties of the efficacy. In the limit of large number of estimating samples, the density $f_M(x)$ of the median estimator tends to an inpulse function

$$\lim_{\frac{m}{n}\to\infty} f_M(x) = \lim_{\frac{m}{n}\to\infty} N\left(M, \frac{\sigma_x^2}{\frac{m}{n}}\right) \qquad (4.2\text{-}19)$$

$$= \delta(x - M)$$

In the same limit, the efficacy becomes

$$\lim_{\frac{m}{n}\to\infty} K(\hat{M}) = \lim_{\frac{m}{n}\to\infty} 4\left[\int \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} d\,F_{\hat{M}}(x)\right]^2 \tag{4.2-20}$$

$$= 4\left[\int \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} \delta(x-M)\,dx\right]^2$$

$$= 4\left[\left.\frac{dG_\theta(M)}{d\theta}\right|_{\theta=0}\right]^2$$

$$= K(M)$$

It is seen from Eq. (4.2-20) that the performance of the detector based on the modified statistic improves as the number of estimating samples increases and, in the limit, the modified median detector is as efficient as the median detector. Thus, the modified detector constitutes a learning system (13) with respect to an unknown, stationary or quasi-stationary median — hence, the name "learning median detector" for the modified detector.

4.2.3 Performance Indices

The modified test statistic is equal to a sum of independent binomially distributed random variables; hence it follows from the central limit theorem (22) that $S_n(\hat{M})$ is asymptotically gaussian under signal and under no-signal conditions. The modified statistic then satisfies condition (A). Condition (D) is fulfilled in the weak signal case investigated here. In the weak signal case, the existence of the efficacy is the only requirement for the statistic to satisfy conditions (B), (C) and (E). The efficacy, given by Eq. (4.2-18), will exist if $\left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0}$ exists — that is, if the median detector efficacy exists. The latter exists for continuous parameter, continuous density functions $g_\theta(x)$. It is seen from Eq. (4.2-11) that $S_n(\hat{M})$ satisfies condition (F) always. The conditions (E) and (F) establish the consistency of $S_n(\hat{M})$.

Since the modified statistic satisfies all of conditions (A)-(F), its performance relation and output signal-to-noise ratio are given by

$$4\left[\int \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_M(x)\right]^2 \bar{\theta}^2 n = 2\left[\mathrm{erf}^{-1}(1-2\alpha) + \mathrm{erf}^{-1}(1-2\beta)\right]^2 \quad (4.2\text{-}21)$$

and

$$\left(\frac{S}{N}\right) = 2\,\bar{\theta}\sqrt{n}\left[\int \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_M(x)\right] \quad (4.2\text{-}22)$$

The efficacy may also be used, as shown in Chapter 2, to obtain the asymptotic relative efficiency of $S_n(\hat{M})$ with respect to a likelihood statistic. Thus, the efficacy $K(\hat{M})$ completely specifies all of the performance indices of the learning median detector for symmetrical first order channel statistics.

4.3 Applications

In the following, the learning median detector is applied to detection problems with symmetrical first order distributions for which it remains distribution-free.

4.3.1 Detection of a Sine Wave in Additive Noise

For this general problem we have

$$\left.\frac{d\,G_\theta(y)}{d\theta}\right|_{\theta=0} = -f_o(y) \qquad (4.3\text{-}1)$$

and using this in Eq. (4.2-18) we obtain

$$K(\hat{M}) = 4\left[\int f_o(x)\, d\, F_M(x)\right]^2 \qquad (4.3\text{-}2)$$

4.3.2 Detection of a Sine Wave in Additive Gaussian Noise

This is a specific case of the previous detection problem where

$$f_o(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty \qquad (4.3\text{-}3)$$

Thus, the efficacy is

$$K(\hat{M}) = 4\left[\int f_o(x)\, f_M(x)\, dx\right]^2 \qquad (4.3\text{-}4)$$

$$= 4\left[\frac{1}{2\pi\sigma_M^{\,2}}\int e^{-\frac{1}{2}x^2\left(1+\frac{1}{\sigma_M^2}\right)}\, dx\right]^2$$

$$= \frac{2}{\pi}\,\frac{1}{(1+\sigma_M^{\,2})}$$

$$= \frac{2}{\pi}\,\frac{1}{(1+\frac{n}{m})}$$

Using Eqs. (3.4-6) and (4.3-4) we obtain the asymptotic relative efficiency of the learning median detector with respect to the median detector. The ARE is

$$ARE_{S_n(\hat{M}),S_n(M)} = \frac{1}{(1+\frac{n}{m})} \qquad (4.3\text{-}5)$$

The asymptotic relative efficiency of the learning median detector with respect to the learning optimum likelihood detector is obtained using Eqs. (4.3-4) and (B-18). It is given by

$$\mathrm{ARE}_{S_n(\hat{M}),L_n^*(\hat{M})} = \frac{2}{\pi} \qquad (4.3\text{-}6)$$

The above result indicates that the learning median and likelihood detectors, both utilizing the sample mean as an estimator of the unknown median, are equally efficient in their learning the median, in the case of gaussian statistics.

4.3.3 Detection of a Sine Wave in Additive Combination of Gaussian and Impulse Noise

This is also a specific case of the general problem discussed in Section 4.3.1. For the present problem $f_o(x)$ is

$$f_o(x) = \frac{1}{\sqrt{2}}\, e^{-\sqrt{2}\,|x|}, \qquad -\infty < y < \infty \qquad (4.3\text{-}7)$$

The efficacy of the learning median detector for large number of estimating samples $\frac{m}{n}$ is given by

$$\begin{aligned} K(\hat{M}) &= 4\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\, e^{-\sqrt{2}\,|x|}\, dF_M(x)\right]^2 \qquad (4.3\text{-}8) \\ &= 4\left[2\int_0^{\infty} \frac{1}{\sqrt{2}}\, e^{-\sqrt{2}\,|x|}\, \frac{1}{\sqrt{2\pi\sigma_M^2}}\, e^{-\frac{x^2}{2\sigma_M^2}}\, dy\right]^2 \\ &= 2\, e^{2\sigma_M^2}\left[1 - \mathrm{erf}\,(\sigma_M)\right]^2 \\ &= 2\, e^{2\frac{n}{m}} \cdot \left[1 - \mathrm{erf}\left(\sqrt{\frac{n}{m}}\right)\right]^2 \end{aligned}$$

Using Eqs. (3.4-8) and (4.3-8) we obtain the asymptotic relative efficiency of the learning median detector with respect to the median detector for known median. The ARE is given by

$$\mathrm{ARE}_{S_n(\hat{M}),S_n(M)} = e^{2\frac{n}{m}}\left[1 - \mathrm{erf}\left(\sqrt{\frac{n}{m}}\right)\right]^2 \qquad (4.3\text{-}9)$$

The asymptotic relative efficiency of the learning median detector, with respect to the learning likelihood detector designed under the gaussian assumption, is obtained utilizing Eqs. (4.3-8) and Eq. (B-18). The ARE is

$$\mathrm{ARE}_{S_n(\hat{M}),L_n^*(\hat{M})} = 2e^{2\frac{n}{m}}\frac{\left[1 - \mathrm{erf}\left(\sqrt{\frac{n}{m}}\right)\right]^3}{\left[1 + \frac{n}{m}\right]} \qquad (4.3\text{-}10)$$

The above expression for ARE indicates that the learning median detector and learning likelihood detector, both utilizing the sample mean as an estimator of the unknown median, are not equally efficient in learning the median of a combination of gaussian and impulse noise. Specifically, it is seen that the learning likelihood detector is more efficient in learning the unknown median than the learning median detector.

4.4 Summary of Results

In this chapter, the statistic on which the median detector is based was modified so that it learns the unknown median. The detector based on the modified statistic was shown to be a learning system with respect to the unknown median, since its performance improves and converges to the performance of the median detector with known median,

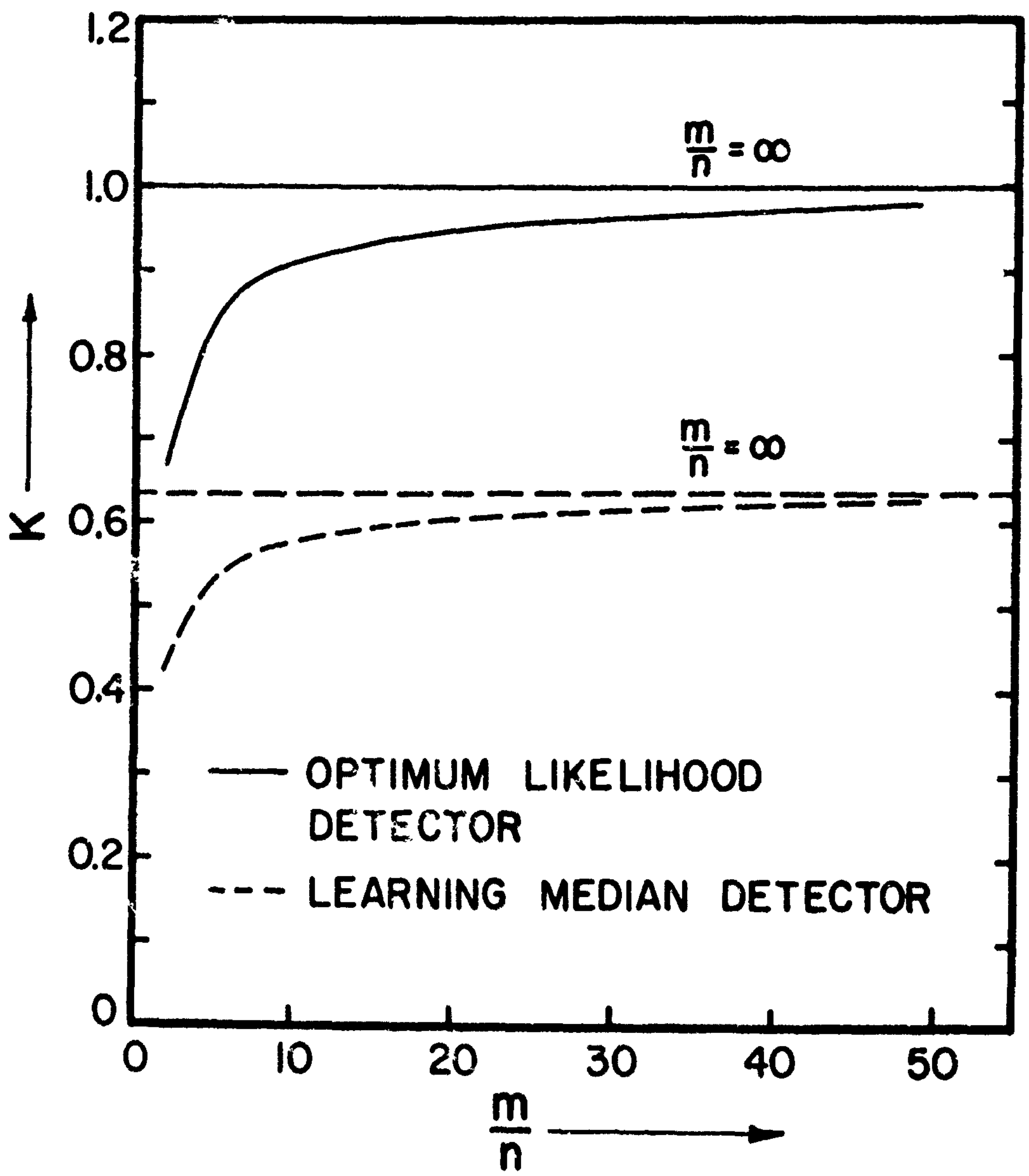

Fig. 9. Efficacy vs. Number of Estimating Samples in the Coherent Detection of a Sine Wave in Gaussian Noise.

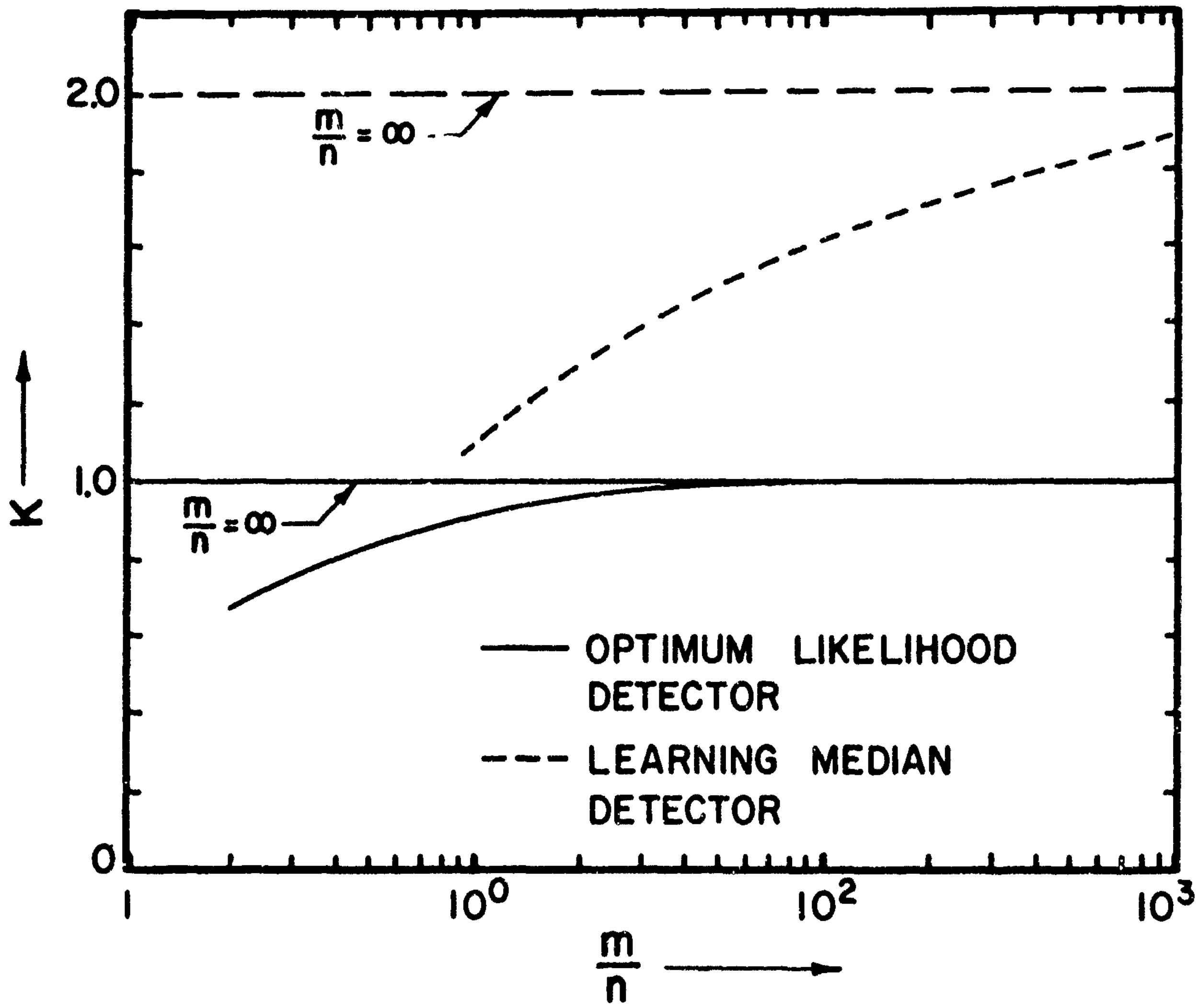

Fig. 10. Efficacy vs. Number of Estimating Samples in the Coherent Detection of a Sine Wave in Impulse and Gaussian Noise.

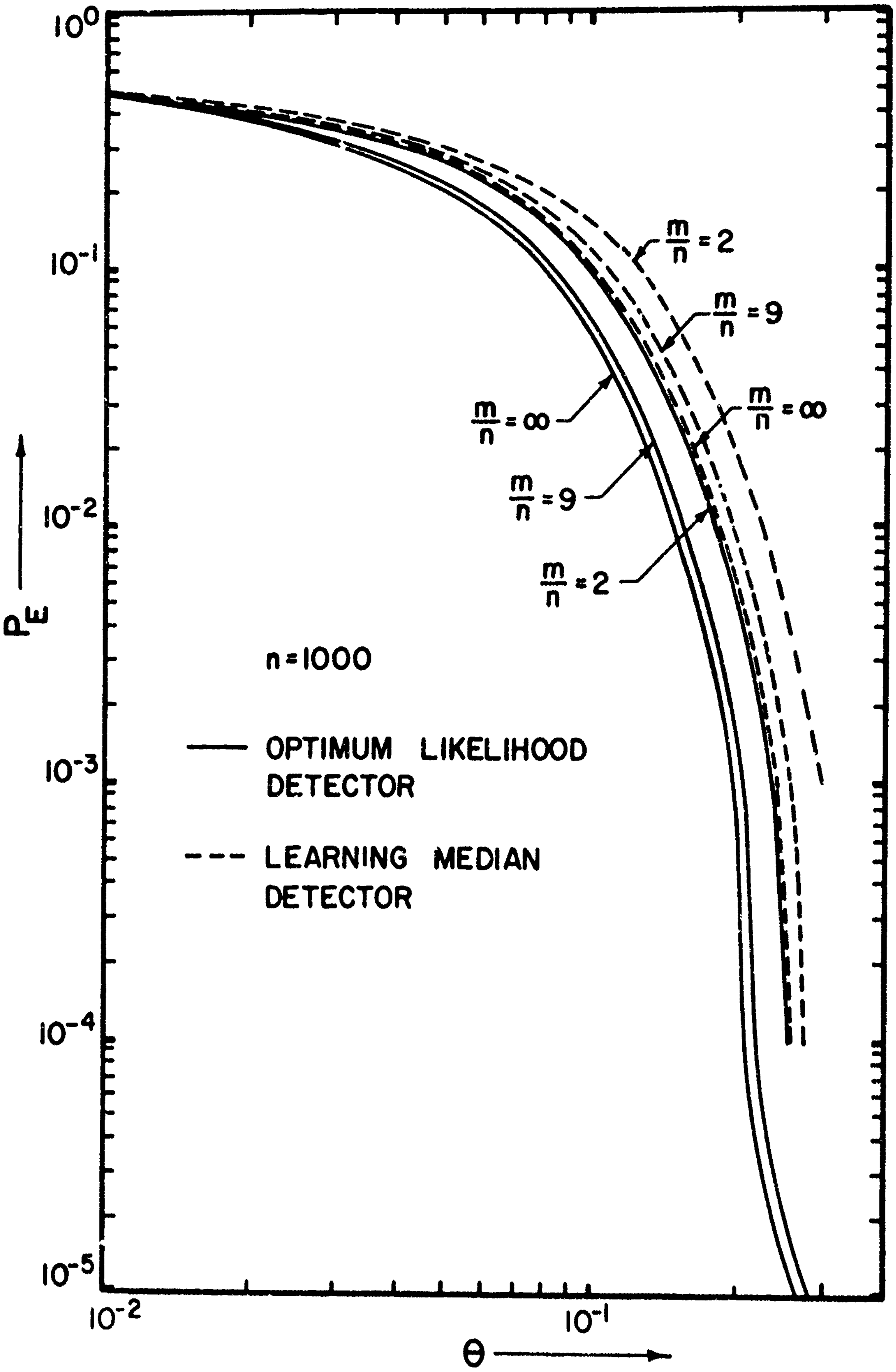

Fig. 11. Probability of Error vs. Received Mean SNR in the Coherent Detection of a Sine Wave in Gaussian Noise.

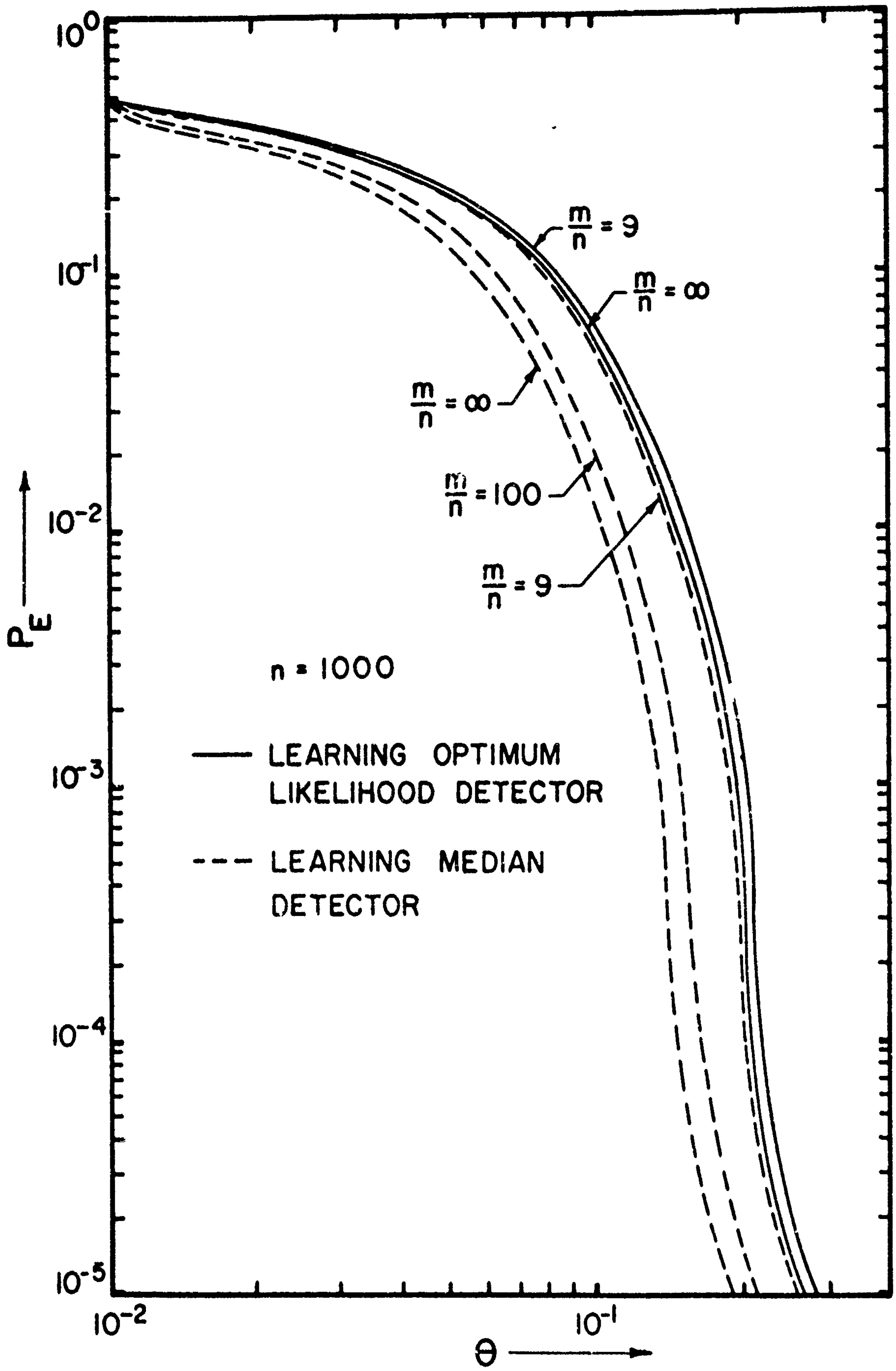

Fig. 12. Probability of Error vs. Received Mean SNR in the Coherent Detection of a Sine Wave in Impulse and Gaussian Noise.

as the learning time increases. The learning median detector was also shown to be distribution-free for a wide class of detection problems — namely, the class with symmetrical first order distributions.

The learning median detector was applied to the detection of a sine wave of known phase in gaussian noise and also in a combination of gaussian and impulse noise. In the gaussian case, the learning median detector and the learning optimum likelihood detector are equally efficient in learning the unknown median. The efficacy of the learning median detector converges rapidly with increasing number of estimating samples to the efficacy of the median detector with known median. These results are presented in Fig. 9. When impulse noise is present in the channel the learning median detector is not as efficient as the learning likelihood detector in learning the unknown median. In this case the learning median detector efficacy does not converge to the median detector efficacy as rapidly as in the detection problem with gaussian noise only. However, in the presence of impulse noise, the information efficiency of the learning median detector is greater than that of the learning likelihood detector even for a moderate number of estimating samples. These results are presented in Fig. 10. A graphical presentation of the functional relation between the learning median detector probability of error and input signal-to-noise ratio is given in Fig. 11 for the case of gaussian channel statistics and in Fig. 12 for a combination of impulse and gaussian channel statistics.

From the results obtained in this chapter, it is concluded that use of the learning median detector instead of the learning likelihood detector entails only a small loss of detection efficiency for gaussian channel statistics; while if impulse noise is present in the channel, use of the learning median detector results in higher detection efficiency. Moreover, the learning median detector is distribution-free for symmetrical first order distributions, hence applicable even when the form of the distributions is unknown.

Chapter 5

ADAPTIVE MEDIAN DETECTOR

5.1 Introduction

The distribution-free coincidence detection procedures investigated in the previous chapters test for the presence of the signal by testing for a change in median under signal and under no-signal conditions. In particular, the median detector is applicable to the detection problem when the median under no-signal conditions is stationary or at most quasi-stationary, and it requires that the value of the median under no-signal conditions be known in order that its false-alarm rate remain distribution-free. The learning median detector does not require knowledge of the median, instead it utilizes an estimate of the unknown median for stationary or at most quasi-stationary medians. However, the learning median detector remains distribution-free only for the class of detection problems with symmetrical first order statistics under no-signal conditions. Thus, use of the learning median detector instead of the median detector is, in effect, equivalent to replacing the restriction of known medians by the restriction of symmetrical first order channel statistics. To summarize, the median and learning median detectors remain distribution-free only when a) the median under no-signal conditions is stationary or at most quasi-stationary and b) when the median is known or when the first order channel statistics are symmetrical under no-signal conditions. However, there exist detection problems in which the location parameters, in particular the median, are non-stationary and, moreover, their time-variations are unknown. In addition, not all detection problems have symmetrical first order statistics. Hence, the need exists for a distribution-free detection procedure applicable ever when the noise medians are unknown and non-stationary and one that remains distribution-free for a wider class of detection problems.

In the present chapter, a modified version of the median detector that is adaptive to rapid changes in the median under no-signal conditions is proposed and investigated. The conditions under which the adaptive median detector remains distribution-free are obtained. It is found that the adaptive median detector remains distribution-free for a much wider class of detection problems than the median detector or the learning median detector. The adaptive median detector is then applied to various detection problems of interest, and its performance is obtained and compared to that of the other distribution-free detectors and to that of comparable likelihood detectors.

5.2 The Modified Test Statistic

The adaptive median detector is based on a modified version of the median detector test statistic. The modified test statistic is

$$S_n = \frac{1}{n} \sum_{i=1}^{n} c(y_i - x_i) \tag{5.2-1}$$

where y_i and x_i, $i = 1, 2, \ldots, n$, are the values of the data and reference samples obtained, respectively, from $Y(t)$ and $N(t)$. The test statistic as defined above is operating on the input waveforms $Y(t)$ and

N(t) in the same manner as the system in Fig. 13.

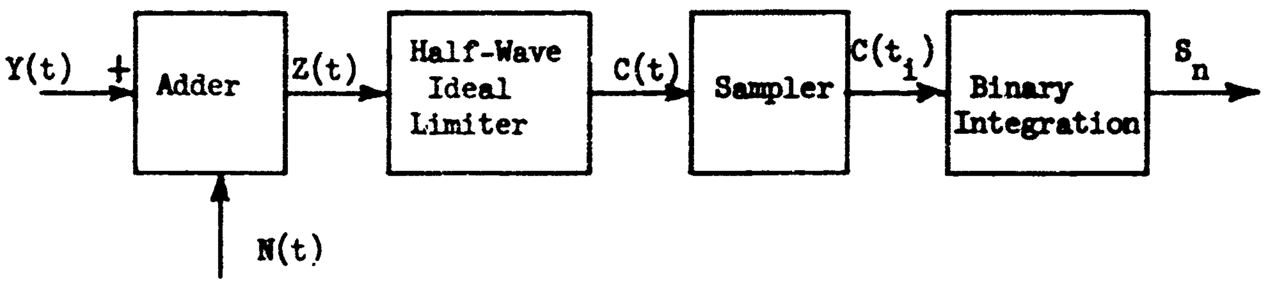

Fig. 13. Block Diagram of Adaptive Median Detector

The reference sample function N(t) is first subtracted from the data sample function Y(t), and the resulting waveform Z(t) is then applied to an ideal half-wave limiter. The output of the limiter is then sampled and the samples averaged to yield S_n.

Application of the adaptive median detector requires, as did the median detectors discussed previously, that the members of each sample pair (Y_i, X_i) have identical medians. This will be true, for instance, if the additive disturbances in the reference and data channels have identical first order statistics. However, for non-stationary channel statistics, the first order statistics, and hence the median, may vary from sample pair to sample pair.

5.2.1 Conditions for Distribution-free Modified Test Statistic

The modified test statistic false-alarm rate will be asymptotically distribution-free, provided the mean and variance of the test statistic are distribution-free under no-signal conditions. The mean and variance under no-signal conditions are

$$E_o[S_n] = \frac{1}{n} \sum_{i=1}^{n} E_o[c(Y_i - X_i)] \tag{5.2-2}$$

$$= \frac{1}{n} \sum_{i=1}^{n} E_o[c(Z_i)]$$

$$= \frac{1}{n} \sum_{i=1}^{n} P[Z_i > 0]$$

$$= \frac{1}{n} \sum_{i=1}^{n} [1 - F_{Z_i}(0)]$$

and

$$\sigma_o^2[S_n] = \frac{1}{n^2} \sum_{i=1}^{n} [1 - F_{Z_i}(0)][F_{Z_i}(0)] \tag{5.2-3}$$

where $Z_i = Y_i - X_i$, and F_{Z_i} is the distribution of Z_i under no-signal conditions. From the above expressions for the mean and variance, it is noted that a necessary and sufficient condition for the modified test statistic to be distribution-free is that zero be a specified quantile of the distribution of Z_i, $i = 1, 2, \ldots, n$, regardless of the channel statistics. In particular, if zero is the median of Z_i, then the distribution of S_n is the same as the distribution of $S_n(M)$ and $S_n(\hat{M})$, under no-signal conditions. A sufficient condition for Z_i to have a zero median is given by the following theorem

<u>Theorem 5.1</u>

For detection problems with reference and data channels possessing symmetrical first order statistics under no-signal conditions, the random variable Z_i, $i = 1, 2, \ldots, n$, has zero median.

Proof:

The proof is the same as that of Theorem 4.1, where one substitutes $f_{o_i}(x)$, the distribution of X_i, $i = 1, 2, \ldots, n$, under no-signal conditions, in place of $f_M(x)$.

The above theorem establishes the distribution-free nature of the adaptive median detector false-alarm rate for the class of detection problems with symmetrical first-order statistics and otherwise arbitrary statistics. In particular, the distribution of Y_i may differ from that of X_i. A wider, and perhaps a more practical, class of detection problems for which the adaptive median detector remains distribution-free is given by the following theorem.

Theorem 5.2

For detection problems with reference and data channels possessing identical first order statistics under no-signal conditions, the random variable Z_i, $i = 1, 2, \ldots, n$, has zero median.

Proof:

Under the conditions of the theorem, $f_{o_i}(x) = g_{o_i}(x)$, thus the value at zero of the distribution of Z_i is

$$F_{Z_i}(o) = P(Z_i > 0) \tag{5.2-4}$$

$$= P(Y_i > X_i)$$

$$= \int_{-\infty}^{\infty} \int_{x}^{\infty} d\, F_{o_i}(x)\, d\, G_{o_i}(y)$$

$$= \int_{-\infty}^{\infty} [1 - F_{o_i}(x)] \, d \, F_{o_i}(x)$$

$$= \frac{1}{2}$$

This completes the proof.

It is seen that theorem 5.2 establishes the distribution-free nature of the adaptive median detector test statistic for the class of detection problems with identical reference and data channel first order statistics.

Thus, it is seen from the above theorems that regardless of whether the channel statistics are stationary or non-stationary, as long as the first order statistics of the reference and data channels are either symmetrical or identical under no-signal conditions, the adaptive median detector false-alarm rate is distribution-free. The mean and variance under no-signal conditions are

$$E_o[S_n] = \frac{1}{n} \sum_{i=1}^{n} [1 - F_{Z_i}(o)] \tag{5.2-5}$$

$$= \frac{1}{2}$$

$$\sigma_o^2 [S_n] = \frac{1}{n^2} \sum_{i=1}^{n} [1 - F_{Z_i}(o)] [F_{Z_i}(o)] \tag{5.2-6}$$

$$= \frac{1}{4n}$$

5.2.2 The Modified Test Statistic Efficacy

The efficacy of the modified statistic will be obtained for the class of detection problems with first order stationary statistics, under no-signal conditions. Under signal conditions, it will be assumed that the distribution of Y_i differs from the distribution of Y_j, $j \neq i$, only through the signal-to-noise ratio. This assumption is satisfied in many detection problems of interest.

For this class of detection problems, the mean and variance under signal conditions are

$$\begin{aligned} E_\theta[S_n] &= \frac{1}{n} \sum_{i=1}^{n} E_{\theta_i}[c(Y_i - X_i)] \qquad (5.2\text{-}7) \\ &= \frac{1}{n} \sum_{i=1}^{n} P[Y_i > X_i] \\ &= \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} [1 - G_{\theta_i}(x)]\, d\, F_o(x) \\ &= \frac{1}{n} \sum_{i=1}^{n} [1 - \int G_{\theta_i}(x)\, d\, F_o(x)] \\ &= \frac{1}{n} \sum_{i=1}^{n} p(\theta_i) \end{aligned}$$

and

$$\begin{aligned} \sigma_\theta^2[S_n] &= \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{\theta_i}^2 [c(Y_i - X_i)] \qquad (5.2\text{-}8) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} p(\theta_i)[1 - p(\theta_i)] \end{aligned}$$

Proceeding as in Chapter 4, we obtain, for the weak signal case and data and reference channels with identical first order statistics under no-signal conditions, that the mean and variance under signal conditions are

$$E_\theta[S_n] = E_o[S_n] - \bar{\theta} \left[\int \frac{d\, G_\theta(x)}{d\theta} \bigg|_{\theta=0} d\, F_o(x) \right] \tag{5.2-9}$$

and

$$\sigma_\theta^{\,2}\, [S_n] = \sigma_o^{\,2}[S_n] - \frac{1}{n}\, \overline{\theta^2} \left[\int \frac{d\, G_\theta(x)}{d\theta} \bigg|_{\theta=0} d\, F_o(x) \right]^2 \tag{5.2-10}$$

where $\bar{\theta}$ and $\overline{\theta^2}$ are, respectively, the mean and mean-square values of the signal-to-noise ratio.

Utilizing Eqs. (5.2-6) and (5.2-9), we obtain the efficacy of S_n

$$K = 4 \left[\int \frac{d\, G_\theta(x)}{d\theta} \bigg|_{\theta=0} d\, F_o(x) \right]^2 \tag{5.2-11}$$

It is shown in the following section that the performance indices are completely specified by the efficacy. Thus, the adaptive nature of the detector may be established by observing the behavior of the efficacy with increasing number of reference samples and fixed number of data samples. It is seen, from Eq. (5.2-11), that the efficacy remains fixed under the above conditions; therefore, the detector is indeed adaptive with respect to an unknown median (13) — hence, the name "adaptive detector" for the modified detector.

5.2.3 Performance Indices

By applying the central limit theorem to S_n, it is seen that the test statistic satisfies condition (A). Condition (D) is fulfilled in the weak signal case investigated here. Eq. (5.2-6) reveals that S_n satisfies condition (F). In the weak signal case and under the conditions for which the efficacy given by Eq. (5.2-11) was derived, the existence of the efficacy insures that the test statistic satisfies conditions (B), (C) and (E). The efficacy will exist if $\left.\frac{d\,G_\theta(x)}{d\theta}\right|_{\theta=0}$ exists — that is, if the median detector efficacy exists. This can be shown as follows. If the median detector efficacy exists, then $\left.\frac{d\,G_\theta(x)}{d\theta}\right|_{\theta=0}$ exists, hence

$$\left.\frac{d\,G_\theta(x)}{d\theta}\right|_{\theta=0} \leq A \tag{5.2-12}$$

where A is a finite number. Thus,

$$K = 4\left[\int \left.\frac{d\,G_\theta(x)}{d\theta}\right|_{\theta=0} d\,F_o(x)\right]^2 \tag{5.2-13}$$

$$\leq 4\left[\int A\,d\,F_o(x)\right]^2$$

$$\leq 4\;A^2$$

and the adaptive median detector efficacy exists also.

Since the modified statistic satisfies all of conditions (A)-(F), its performance relation and output signal-to-noise ratio for the weak signal case and data and reference channels with identical, stationary first order statistics, under no-signal conditions, are given by

$$4\left[\int \frac{d\,G_\theta(x)}{d\theta}\bigg|_{\theta=0} d\,F_o(x)\right]^2 \bar{\theta}^2 n = 2\left[\mathrm{erf}^{-1}(1-2\alpha) + \mathrm{erf}^{-1}(1-2\beta)\right]^2 \tag{5.2-14}$$

and

$$\left(\frac{S}{N}\right) = 2\,\bar{\theta}\sqrt{n}\left[\int \frac{d\,G_\theta(x)}{d\theta}\bigg|_{\theta=0} d\,F_o(x)\right] \tag{5.2-15}$$

The efficacy may also be used, as shown in Chapter 2, to obtain the asymptotic relative efficiency of S_n with respect to other detectors. Thus, the efficacy K completely specifies all of the performance indices of the adaptive median detector for the conditions for which the efficacy given in Eq. (5.2-11) is applicable.

5.3 Applications

In the following, the median detector is applied to specific detection problems; its performance in the problems is evaluated and compared to that of other distribution-free detectors and to that of comparable likelihood detectors.

5.3.1 Detection of a Sine Wave of Known Phase in Additive Noise-General Case

For this general detection problem we have

$$\frac{d\,G_\theta(x)}{d\theta}\bigg|_{\theta=0} = -\,f_o(x) \tag{5.3-1}$$

and using this in Eq. (5.2-11) we obtain

$$K = 4 \left[\int f_o^{\,2}(x)\, dx \right]^2 \tag{5.3-2}$$

The efficacy of the likelihood detector, appropriate to the problem, is given by Eq. (B-33) as

$$K^* = \frac{1}{2} \tag{5.3-3}$$

Thus, the asymptotic relative efficiency of the adaptive median detector with respect to the comparable likelihood detector is

$$ARE_{S_n, L_n^*} = 8\left[\int f_o^{\,2}(x)\, dx \right]^2 \tag{5.3-4}$$

Pitman (29) has shown that the lowest possible value of the above integral squared is equal to 9/125. Thus, for this general problem, the lower bound of the adaptive median detector efficacy is

$$K_{min.} = 0.288 \tag{5.3-5}$$

and the lower bound of the asymptotic relative efficiency is

$$ARE_{S_n, L_n^*} = 0.576 \tag{5.3-6}$$

The asymptotic relative efficiency may be anything from the minimum given above to infinity, depending on $f_o(x)$.

5.3.2 Detection of a Sine Wave of Known Phase in Additive Gaussian Noise

This problem is a specific case of the previous general detection problem , with $f_o(x)$ given by Eq. (A-4). Utilizing Eq. (A-4) in Eq. (5.3-2), we obtain

$$K = 4\left[\int \frac{1}{2\pi} e^{-x^2} dx\right]^2 \tag{5.3-7}$$

$$= \frac{1}{\pi}$$

Using Eqs. (5.3-7) and (5.3-3) we obtain the asymptotic relative efficiency of the adaptive median detector with respect to the adaptive optimum likelihood detector. The ARE is

$$ARE_{S_n, L_n^*} = \frac{2}{\pi} \tag{5.3-8}$$

The asymptotic relative efficiency of the adaptive median detector with respect to the median detector is obtained using Eqs. (5.3-7) and (3.4-6). It is given by

$$ARE_{S_n, S_n(M)} = \frac{1}{2} \tag{5.3-9}$$

The asymptotic relative efficiency of the adaptive median detector with respect to the learning median detector is derived using Eqs. (5.3-7) and (4.3-4). The ARE is

$$ARE_{S_n, S_n(\hat{M})} = \frac{1}{2}\left(1 + \frac{n}{m}\right) \tag{5.3-10}$$

It is seen from Eq. (5.2-24) that use of the adaptive median detector instead of the median detector results in reduction of the information rate by one-half. Use of the adaptive median detector instead of the learning median detector entails a loss in information rate even for a small number of estimating samples, such as two or three. For estimating sample sizes greater than ten, the information rate is almost halved.

5.3.3 Detection of a Sine Wave of Known Phase in Additive Gaussian and Impulse Noise

This problem also is a specific case of the general problem. Thus, using Eq. (A-6) in Eqs. (5.3-2) and (5.3-4), we obtain for c = 1

$$K = 4\left[2\int_0^{\infty} a^2 e^{-2bx} dx\right]^2 \tag{5.3-11}$$

$$= \frac{1}{2}$$

and

$$ARE_{S_n, L_n^*} = 1 \tag{5.3-12}$$

The asymptotic relative efficiencies of the adaptive median detector with respect to the median and learning median detectors are obtained utilizing Eq. (5.3-11) and Eqs. (3.4-8) and (4.38), respectively. The ARE's, for c = 1, are

$$ARE_{S_n, S_n(M)} = \frac{1}{4} \tag{5.3-13}$$

and

$$ARE_{S_n, S_n(\hat{M})} = \frac{1}{4} e^{-2\frac{n}{m}} \left[1 - erf\left(\sqrt{\frac{n}{m}}\right)\right]^2 \tag{5.3-14}$$

An examination of the above symptotic relative efficiencies reveals that in the case of gaussian and impulse noise, the adaptive median detector is as efficient as the comparable likelihood detector. However, the adaptive median detector information rate is only 1/4 that of the median detector. Even for small estimating sample sizes, the information rate of the adaptive median detector is smaller than that of the learning median detector.

5.3.4 Detection of a Sine Wave of Unknown Phase in Additive Gaussian Noise

Using Eq. (A-14) in Eq. (5.2-11), we obtain

$$K = 4\left[\int \frac{d\,G_\theta(x)}{d\theta}\bigg|_{\theta=0} d\,F_o(x)\right]^2 \tag{5.3-15}$$

$$= 4\left[\int x\,\phi^2\,(0,\,\sigma)\,dx\right]^2$$

$$= 0$$

The comments made in Chapter 3 regarding this problem are applicable here also.

5.3.5 Envelope Detection of a Sine Wave in Narrow-Band Gaussian Noise

Using Eq. (A-19) in Eq. (5.2-11), we obtain

$$K = 4\left[\int_0^\infty \frac{x^3}{2\sigma^4}\, e^{-\frac{x^2}{\sigma^2}}\, dx\right]^2 \tag{5.3-16}$$

$$= 0.25$$

Using Eqs. (5.3-16) and (3.4-15), we obtain the asymptotic relative efficiency of the adaptive median detector with respect to the median detector. The ARE is

$$ARE_{S_n, S_n(M)} = 0.52 \tag{5.3-17}$$

The asymptotic relative efficiency of the adaptive median detector with respect to the comparable likelihood detector is obtained using Eqs. (5.3-16) and (B-48). The ARE is

$$\mathrm{ARE}_{S_n, L_n^*} = 1 \tag{5.3-18}$$

5.3.6 Square-Law Detection of a Sine Wave in Narrow-Band Additive Gaussian Noise

Using Eq. (A-23) in Eq. (5.2-11), we obtain

$$K = 4\left[\int_0^\infty y\, e^{-2y}\, dy\right]^2 \tag{5.3-19}$$

$$= 0.25$$

and the asymptotic relative efficiencies are as in the previous problem.

5.4 Summary of Results

In this chapter, a modified version of the median detector that is adaptive to an unknown stationary or non-stationary median was proposed and investigated. The conditions under which the adaptive median detector remains distribution-free were also obtained. It was shown that the adaptive median detector remains distribution-free for two wide classes of detection problems. Specifically, the adaptive median detector false-alarm rate remains distribution-free for all detection problems with symmetrical first-order statistics under no-signal conditions. It also remains distribution-free for all detection problems with identical first-order reference and data channel statistics, under no-signal conditions.

The adaptive median detector was applied to the detection of a sine wave in additive noise, and its performance in the problem investigated. The results of this investigation are presented in Table 1.

Table 1

Performance of Adaptive Median Detector in Detecting a Sine Wave in Additive Noise

	Sine Wave of Known Phase			Sine Wave of Unknown Phase Gaussian Noise	
	Lower Bound	c = 2	c = 1	Predetection Processing	No Predetection Processing
K	0.288	0.318	0.500	0.250	0
ARE_{S_n, L_n}	0.576	0.637	1.000	1.000	0
$ARE_{S_n, S_n(M)}$		0.500	0.250	0.520	
$ARE_{S_n, S_n(\hat{M})}$		$\frac{1}{2}\left(1+\frac{n}{m}\right)$	$\frac{1}{4}e^{-2\frac{n}{m}}\left[1-\mathrm{erf}\left(\sqrt{\frac{n}{m}}\right)\right]^{-2}$		

An examination of the above table reveals that the adaptive median detector is highly efficient for the detection of a sine-wave in additive noise of unknown median. Specifically, the adaptive median detector information rate is never less than 57.6% of the information rate of the comparable likelihood detector. In the case of a sine wave of known phase and guassian noise, the adaptive median detector

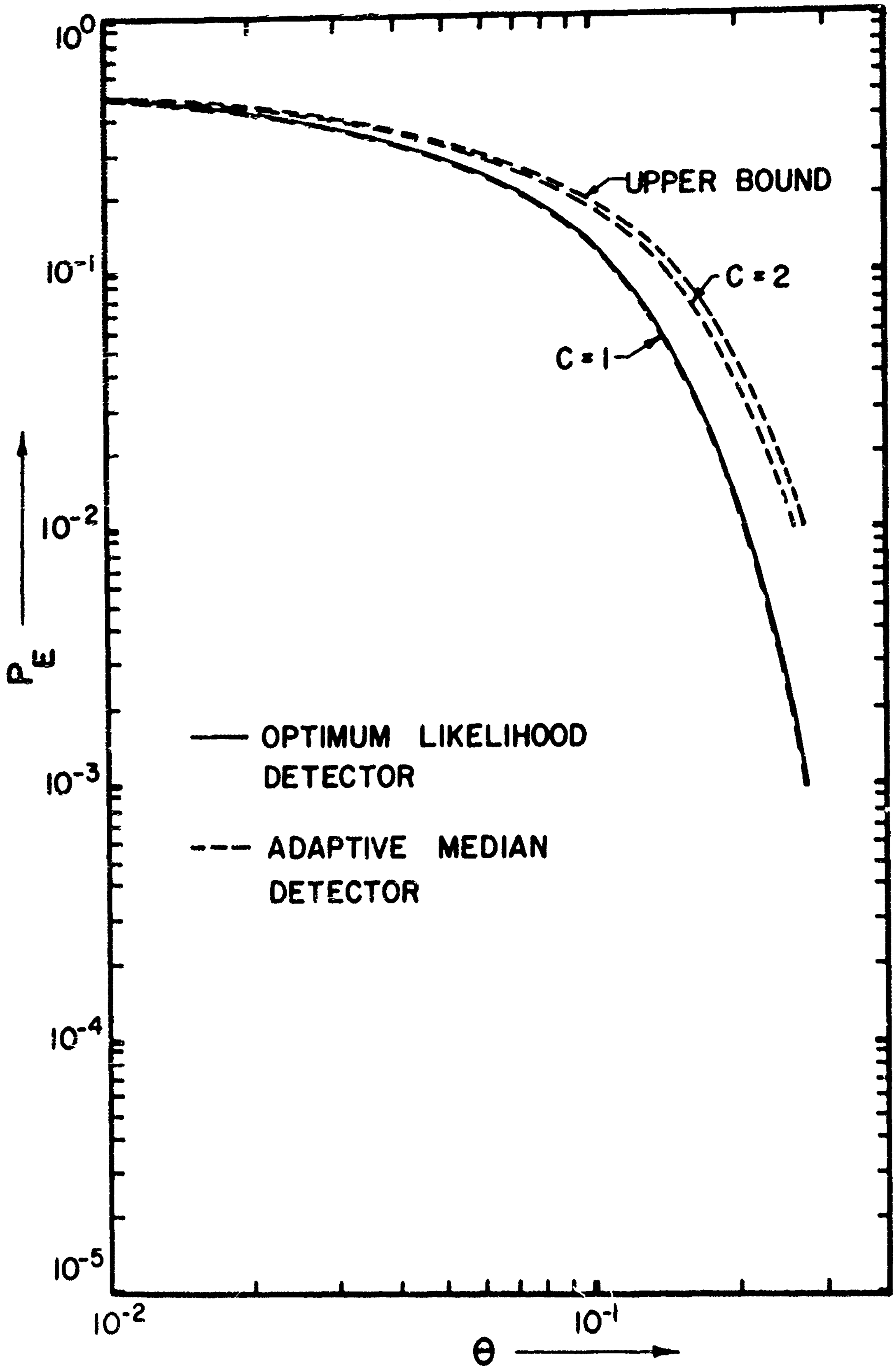

Fig. 14. Probability of Error vs. Received Mean SNR in the Coherent Detection of a Sine Wave in Additive Noise.

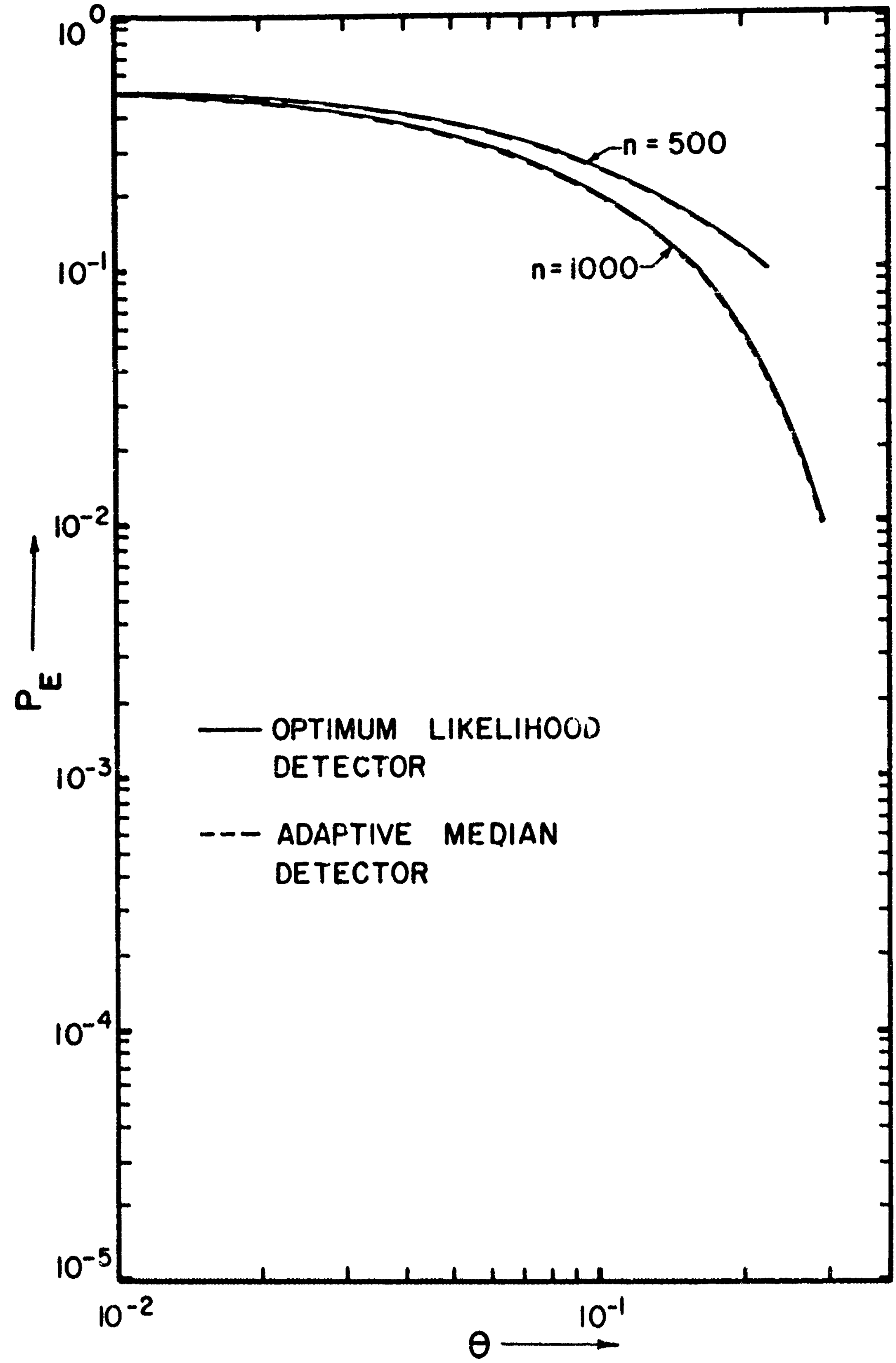

Fig. 15. Probability of Error vs. Received Mean SNR in the Non-coherent Detection of a Sine Wave in Gaussian Noise.

information rate is 63.7% of that of the optimum likelihood detector. However, when the noise is a combination of impulse and gaussian noise, the adaptive median detector information rate is equal to the information rate of the comparable likelihood detector. For the detection problem of a sine wave of unknown phase in guassian noise and with predetection processing of the input waveform, the adaptive median detector information rate is equal to that of the adaptive "optimum" likelihood detector.

It is also seen from Table 1 that the adaptive median detector is less efficient than either the median detector or the learning median detector. This, however, is expected since the median detector requires and uses knowledge about the channel statistics, namely the value of the median of the additive noise, that the adaptive median detector does not require and does not use. The learning median detector, although it does not require this additional knowledge of the channel statistics, does require and utilize more reference samples than the adaptive median detector.

In Figs. 14 and 15, the probability of error of the adaptive median detector is plotted vs. the input signal-to-noise ratio for all the detection problems investigated in this chapter.

From the results obtained in this chapter, it is concluded that use of the adaptive median detector instead of an adaptive likelihood detector entails either a small loss of detection efficiency or none at all. Moreover, the adaptive median detector is distribution-free for wide classes of detection problems, hence applicable even when the form of the distributions is unknown.

Chapter 6

THE T-DETECTOR

6.1 Introduction

In this chapter, a distribution-free detector of stochastic signals in noise is proposed and investigated. This is based on a test statistic that tests for the presence of the signal by testing for a difference in variance between the reference and data samples. The test statistic is the so-called T-statistic—hence the name "T-detector" for the distribution-free detector that utilizes it. In the following, the general properties of the T-statistic are given and its efficacy, output signal-to-noise ratio, and performance relation are obtained. Subsequently, the T-detector is applied to the detection of a gaussian signal in gaussian noise, and its performance is evaluated and compared to that of the optimum detector.

6.2 The T statistic

The T-detector is based on the T-statistic first proposed by Sukhatme (31) and defined as

$$T_{mn} = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \psi(y_i, x_j) \tag{6.2-1}$$

where

$$\psi(y_i, x_j) = 1, \quad \text{if} \quad \begin{array}{l} 0 < x_j < y_i \\ y_i < x_j < 0 \end{array}$$

$$= 0, \quad \text{otherwise}$$

Application of the T-statistic to the detection problem is based on the assumption that the variances of the reference and data samples are the same under no-signal conditions. This will be true, for instance, if the additive disturbances in the reference and data channels have identical first order statistics under no-signal conditions. In this investigation, the above condition on the first order statistics will be assumed. Moreover, the reference and data channel first order statistics will be assumed stationary or at most quasi-stationary. Thus, the random variables Y_i, $i = 1, 2, \ldots, n$ will have identical first order distribution functions, as will the random variables X_j, $j = 1,2,\ldots, m$.

6.2.1 Conditions for Distribution-free Test Statistic

The T-statistic is a modified version of the Wilcoxon-Mann-Whitney (31) statistic. Mann and Whitney proved (4) the asymptotic nromality of the Wilcoxon statistic under no-signal conditions and Lehman proved it (4) under signal conditions. Utilizing these results, it can be shown that the T-statistic is asymptotically normally distributed under signal and under no-signal conditions. Thus, the T-statistic false-alarm rate will be asymptotically distribution-free if its mean and variance are distribution-free under no-signal conditions. The mean and variance under no-signal conditions are (31)

$$E_0[T_{mn}] = \int_0^{\infty} [1 - F_o(x)]\, d\, F_o(x) + \int_{-\infty}^{0} F_o(x)\, d\, F_o(x) \qquad (6.2\text{-}2)$$

$$= F_o(x) \Big|_0^\infty - \frac{1}{2} F_o^2(x) \Big|_0^\infty + \frac{1}{2} F_o^2(x) \Big|_{-\infty}^0$$

$$= F_o^2(o) - F_o(o) + \frac{1}{2}$$

and

$$\sigma_o^2[T_{mn}] = \frac{1}{mn}\Big[\int_0^\infty F_o \, d\, F_o - \int_{-\infty}^0 F_o \, dF_o + \tag{6.2-3}$$

$$+ (n-1)\left\{ \int_0^\infty (1 - F_o)^2 \, dF_o + \int_{-\infty}^0 F_o^2 \, dF_o \right\} +$$

$$+ (m-1)\left\{ \int_{-\infty}^\infty F_o^2 \, dF_o - \int_{-\infty}^\infty F_o \, dF_o + \frac{1}{4} \right\} +$$

$$- (m+n-1)\left\{ \int_0^\infty F_o \, dF_o - \int_{-\infty}^0 F_o \, dF_o \right\}^2 \Big]$$

$$= \frac{1}{mn}\Big[(1 - m - n) F_o^4(o) + (m + 2n - 3) F_o^2(o) +$$

$$+ (1-n) F_o(o) + \frac{n - 2m + 4}{12} \Big]$$

where $F_o(x)$ is the distribution of the reference and data samples under no-signal conditions. From the above expressions, it is seen that a necessary and sufficient condition for the T-statistic false-alarm rate to be distribution-free is that zero be a specified quantile of the distribution $F_o(x)$, regardless of the channel statistics. Thus, the T-statistic false-alarm rate will be distribution-free for the class of distribution functions $F_o(x)$ with zero median. For those distribution functions with non-zero medians, the medians may be subtracted out from the incoming reference and data sample functions so that samples obtained from the modified sample functions have zero median. In the latter case, the T-statistic becomes

$$T_{mn}(M,N) = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \psi(y_i', x_j') \qquad (6.2\text{-}4)$$

where

$$y_i' = y_i - M \qquad (6.2\text{-}5)$$

$$x_j' = x_j - N$$

and M and N are the medians of the data and reference samples under no-signal conditions. Thus, Y_i', $i = 1, 2, \ldots, n$ and Y_j', $j = 1, 2, \ldots, m$ have zero medians.

To summarize, the T-statistic false-alarm rate will be distribution-free a) for the class of detection problems with zero medians and b) for the class of detection problems with non-zero medians provided these medians are known. With regard to the latter class of detection problems, it must be pointed out that there exists a subclass that does not require knowledge of the medians under no-signal conditions. This is the class of detection problems with symmetrical first-order statistics under no-signal conditions. The mean and median for this class of problems coincide. Thus, the value of the non-zero medians is not required since the medians can be made zero by subjecting the reference and data channel sample functions to capacitive filtering prior to their examination by the T-detector.

For the case of input reference and data waveforms with zero medians, the mean and variance of the T-statistic under no-signal conditions, obtained from Eqs. (6.2-2) and (6.2-3), are

$$E_o[T_{mn}(M,N)] = \frac{1}{4} \tag{6.2-6}$$

and

$$\sigma_o^2[T_{mn}(M,N)] = \frac{m+n+7}{48\ mn} \tag{6.2-7}$$

$$= \frac{1}{48n} \qquad \text{for } m >> n >> 7$$

6.2.2 <u>The T-Statistic Efficacy</u>

For stationary or quasi-stationary first-order statistics under signal and under no-signal conditions and for identical reference and data channel first-order statistics under no-signal conditions, the T-statistic mean and variance under signal conditions are (31)

$$E_\theta[T_{mn}(M,N)] = \int_0^\infty [1 - G_\theta(x)]\ dF_o(x) + \int_{-\infty}^0 G_\theta(x)\ dF_o(x) \tag{6.2-8}$$

and

$$\begin{aligned}\sigma_\theta^2[T_{mn}] = \frac{1}{mn}\Bigg[&\int_0^\infty F_o(x)\ dG_\theta(x) - \int_{-\infty}^0 F_o(x)\ dG_\theta(x) + \\ &+ (n-1)\left\{\int_0^\infty [1 - G_\theta(x)]^2 dF_o(x) + \int_{-\infty}^0 G_\theta^2(x)\ dF_o(x)\right\} + \\ &+ (m-1)\left\{\int_{-\infty}^\infty F_o^2(x)\ d\,G_\theta(x) - \int_{-\infty}^\infty F_o(x)\ dG_\theta(x) + \frac{1}{4}\right\} \\ &- (m+n-1)\left\{\int_0^\infty F_o(x)\ dG_\theta(x) - \int_{-\infty}^0 F_o(x)\ dG_\theta(x)\right\}^2\Bigg]\end{aligned} \tag{6.2-9}$$

Applying the mean value theorem (23) to $G_\theta(y)$, we obtain for the weak signal case

$$G_\theta(y) - G_o(y) = \theta\left[\left.\frac{d\,G_\theta(y)}{d\theta}\right|_{\theta=0}\right] \tag{6.2-10}$$

or, since $G_o(y) = F_o(y)$, we have

$$G_\theta(y) - F_o(y) = \theta\left[\left.\frac{d\,G_\theta(y)}{d\theta}\right|_{\theta=0}\right] \tag{6.2-11}$$

Substituting this in Eq. (6.2-8) we obtain

$$E_\theta[T_{mn}(M,N)] = \int_0^\infty [1 - F_o(x)]\,d\,F_o(x) + \int_{-\infty}^0 F_o(x)\,dF_o(x) +$$

$$\theta\left[\int_{-\infty}^0 \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x) - \int_0^\infty \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x)\right]$$

$$= E_o[T_{mn}(M,N)] + \theta\left[\int_{-\infty}^0 \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x) - \right.$$

$$\left. - \int_0^\infty \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x)\right] \tag{6.2-12}$$

Using Eqs. (6.2-7) and (6.2-12), we obtain the efficacy of the T-statistic for $m >> n$. This is

$$K(M,N) = 48\left[\int_{-\infty}^0 \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x) - \int_0^\infty \left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x)\right]^2 \tag{6.2-13}$$

6.2.3 <u>Performance Indices</u>

As stated previously, the T-statistic is asymptotically normally distributed under signal and under no-signal conditions; hence, it satisfies condition (A). Condition (D) is fulfilled in the weak signal case investigated here. Eq. (6.2-7) reveals that $T_{mn}(M,N)$ satisfied condition (F). In the weak signal case and under the conditions for which the efficacy given by Eq. (6.2-13) was derived, the existence of the efficacy insures that the test statistic satisfies conditions (B), (C) and (E). The efficacy will exist if $\left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0}$ exists. The latter becomes apparent from an examination of Eq. (6.2-13).

If $\left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0}$ exists, then the test statistic satisfies all of conditions (A)-(F) in the weak signal case. Hence, for data and reference channels with first-order statistics identical under no-signal conditions and stationary or quasi-stationary both under signal and under no-signal conditions, the T-statistic performance relation and output signal-to-noise ratio are given by

$$48\left[\int_{-\infty}^{0}\left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x) - \int_{0}^{\infty}\left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x)\right]^2 \bar{\theta}^2 n =$$

$$= 2\left[\mathrm{erf}^{-1}(1-2\alpha) + \mathrm{erf}^{-1}(1-2\beta)\right]^2 \qquad (6.2\text{-}14)$$

and

$$\left(\frac{S}{N}\right) = \bar{\theta}\sqrt{48n}\left[\int_{-\infty}^{0}\left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x) - \int_{0}^{\infty}\left.\frac{dG_\theta(x)}{d\theta}\right|_{\theta=0} dF_o(x)\right] \qquad (6.2\text{-}15)$$

The efficacy may also be used to obtain the asymptotic relative efficiency of $T_{mn}(M,N)$ with respect to other statistics. Thus, the efficacy specifies all of the performance indices of the T-detector for the conditions under which the efficacy given in Eq. (6.2-13) is applicable.

6.3 Applications

In the following, the T-detector is applied to the detection of a gaussian signal in gaussian noise; its performance is evaluated and compared to the performance in the same problem of the optimum likelihood detector. Results concerning the asymptotic relative efficiency of the T-detector with respect to a particular likelihood detector, in the general problem of scalar alternatives, are also given.

6.3.1 Detection of Narrow-Band White Gaussian Signal in Additive White Gaussian Noise

Using Eqs. (A-25) and (A-28) in Eq. (6.2-13) we obtain the T-statistic efficacy. This is

$$K(M,N) = 48\left[\int_0^{\infty} \frac{x}{2}\ \phi^2(x)\ dx - \int_{-\infty}^{0} \frac{x}{2}\ \phi^2(x)\ dx\right]^2 \qquad (6.3\text{-}1)$$

$$= \frac{3}{\pi^2}$$

Using Eqs. (6.3.1) and (B-59) we obtain the asymptotic relative efficiency of the T-detector with respect to the optimum likelihood detector. The ARE is

$$ARE_{T_{mn}(M,N),\ L_n^*} = 0.61 \qquad (6.3\text{-}2)$$

6.3.2 T-statistic Performance in the General Problem of Scalar Alternatives

The problem of scalar alternatives is one with distributions $F_o(x)$ and $G_\theta(x)$, under the hypothesis and under the alternative, respectively, related as follows

$$G_\theta(x) = F_o(\theta\, x) \tag{6.3-3}$$

Sukhatme (31) has obtained in general the asymptotic relative efficiency of the T-statistic with respect to the variance-ratio F-test, a likelihood statistic optimum for gaussian statistics. The asymptotic relative efficiency for the problem of scalar alternatives is given by (31)

$$\mathrm{ARE}_{T_{mn}(M,N),\ F} = 12\,(\beta_2-1)\left[\int_0^\infty x\, f_o^{\,2}(x)\, dx - \int_{-\infty}^0 x\, f_o^{\,2}(x)\, dx\right]^2 \tag{6.3-4}$$

where

$$\beta_2 = \frac{\int [x-E(X)]^4 dF_o(x)}{\left\{\int [x-E(X)^2\, dF_o(x)\right\}^2} \tag{6.3-5}$$

It can be seen from Eq. (6.3-4) that the asymptotic relative efficiency can be anything from zero to infinity, depending on $f_o(x)$. In particular if $f_o(x) = \frac{1}{2} e^{-|x|}$, the ARE is equal to 0.94.

6.4 Summary of Results

In this chapter, the T-detector for the detection of stochastic signals in noise was proposed and investigated. It was shown that the T-detector false-alarm rate can be made distribution-free given the medians of the reference and data samples under no-signal conditions. Even in the absence of this minimal information concerning the statistics of the detection problem, the T-detector false-alarm rate was shown to be distribution-free for two classes of detection problems, a) the class

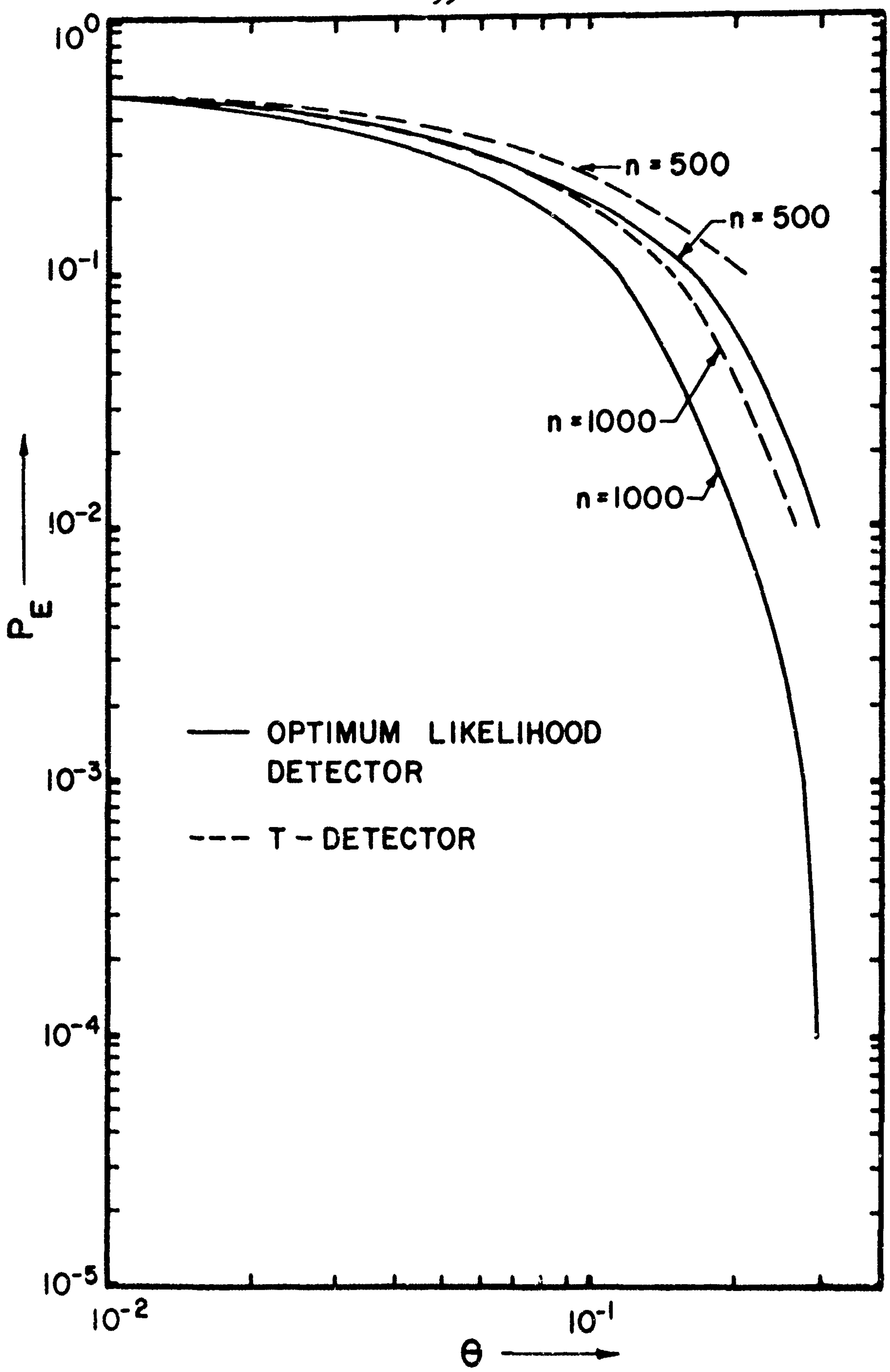

Fig. 16. Probability of Error vs. Received Mean SNR in the Detection of a Gaussian Signal in Gaussian Noise.

of detection problems with zero medians under no-signal conditions, and b) the class of detection problems with symmetrical first-order statistics under no-signal conditions.

The T-detector was applied to the detection of a gaussian signal in gaussian noise, and its performance in the problem investigated. It was found that the T-detector is reasonably efficient for gaussian statistics and highly efficient for some non-gaussian statistics. Specifically, the T-detector information rate for the case of a gaussian signal in gaussian noise was shown to be 61% of that of the optimum likelihood detector. The results for this problem are presented graphically in Fig. 16.

From the results obtained here, it is concluded that use of the T-detector instead of a likelihood detector entails only a small loss of detection efficiency for gaussian channel statistics; while for non-gaussian statistics, an increase in efficiency is possible, depending on $f_o(x)$. Moreover, the T-detector is distribution-free for wide classes of detection problems, hence applicable even when the form of the distributions is unknown.

Chapter 7

ADAPTIVE T-DETECTOR

7.1 Introduction

The T-detector investigated in the previous chapter can be made distribution-free provided the medians of the reference and data samples under no-signal conditions are known. The T-detector remains distribution-free even when the medians are unknown but only for two limited classes of detection problems, namely, the class of detection problems with zero medians and the class of problems with symmetrical first-order reference and data channel statistics under no-signal conditions. However, the above classes do not include many of the problems of practical importance. There exist problems in which the location parameters of the distributions, in particular the medians, are non-stationary with unknown time variations. In addition, not in all detection problems are the first order statistics symmetrical or the medians zero. Hence, the need exists for a distribution-free detection procedure which is applicable even when the noise medians are changing and/or unknown and one that remains distribution-free for a wider class of detection problems.

In this chapter, a modified version of the T-detector that is adaptive to rapid changes in the medians and/or to unknown medians is proposed and investigated. The conditions under which the adaptive T-detector remains distribution-free are obtained. It is found that the adaptive T-detector remains distribution-free for a much wider class of detection problems than the T-detector. The adaptive T-detector is then applied to the detection of a gaussian signal in gaussian noise, and its performance is evaluated and compared to the performances of the T-detector and of the optimum likelihood detector.

7.2 The Modified Test Statistic

The adaptive T-detector is based on a modified version of the T-statistic. The modified T-statistic is

$$T_{mn} = \frac{4}{mn} \sum_{i=1}^{n/2} \sum_{j=1}^{m/2} \psi[(y_{2i} - y_{2i-1}), (x_{2j} - x_{2j-1})] \qquad (7.2\text{-}1)$$

$$= \frac{4}{mn} \sum_{i=1}^{n/2} \sum_{j=1}^{m/2} \psi(v_i, u_j)$$

where $v_i = y_{2i} - y_{2i-1}$, $u_j = x_{2j} - x_{2j-1}$, and y_k, $k = 1,2,\ldots, n$, and x_ℓ, $\ell = 1,2,\ldots, m$, are the values of the data and reference samples obtained, respectively, from $Y(t)$ and $N(t)$. The function $\psi(v,u)$ was defined previously. The test statistic as defined above is operating on the sample functions $Y(t)$ and $N(t)$ in the same manner as the system shown in Figure 17. Here $1/\tau_1$ and $1/\tau_2$ are, respectively, the rates at which $Y(t)$ and $N(t)$ are sampled.

Application of the adaptive T-detector to the detection of stochastic signals in noise requires, as did the T-detector, that the variances of the reference and data samples are identical under no-signal conditions. The latter will be true, for instance, if the additive disturbances in the reference and data channels have first order statistics of identical form and differing, if at all, only in their location parameters.

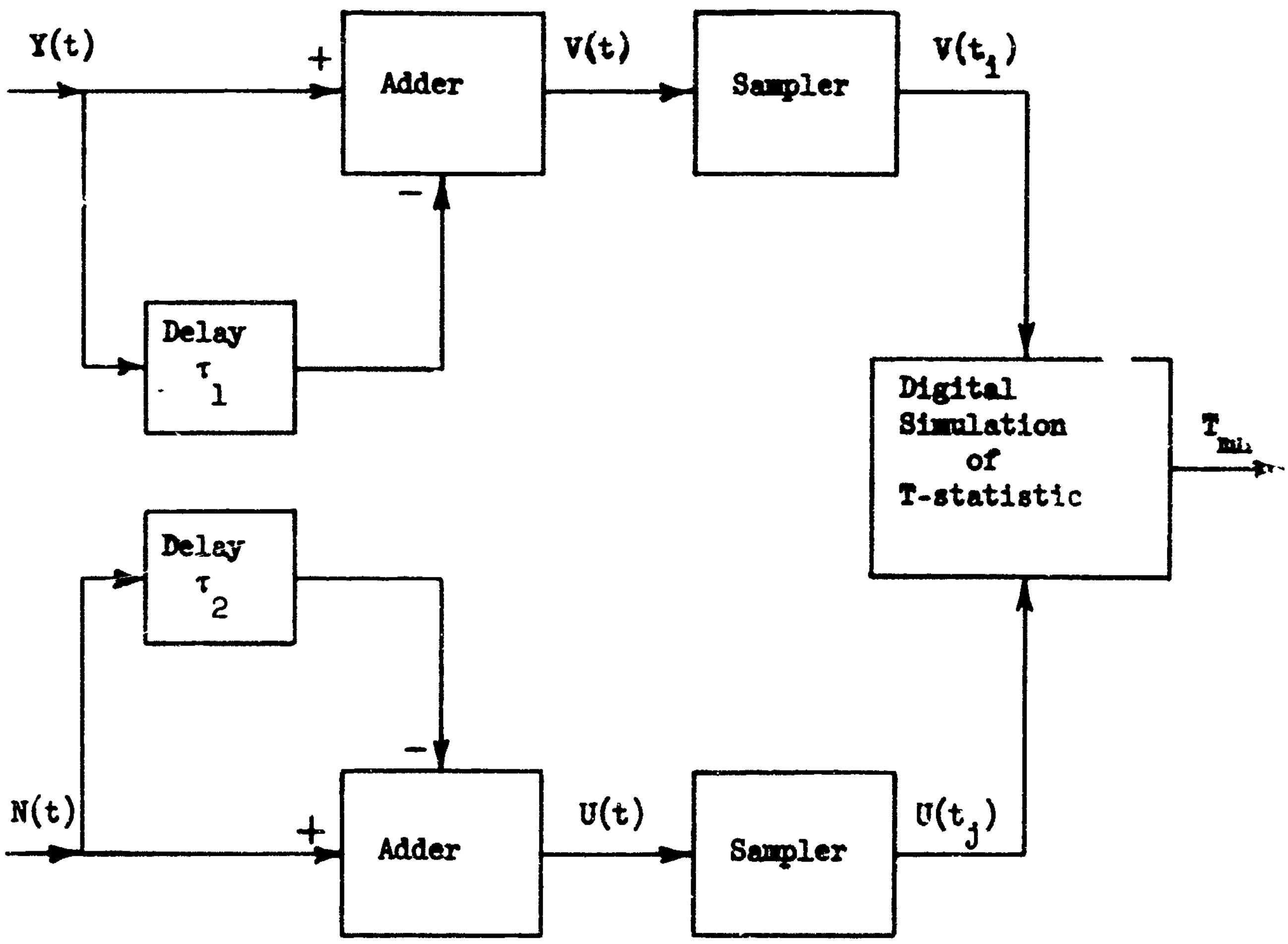

Fig. 17. Block Diagram of Adaptive T-detector

7.2.1 <u>Conditions for Distribution-free Modified Test Statistic</u>

The conditions under which the modified T-statistic is distribution-free are given by the following theorem.

<u>Theorem 7.1</u>

The modified T-statistic false alarm rate remains asymptotically distribution-free for the class of detection problems with reference and data channel first-order statistics having the following properties under no-signal conditions:

a) the first-order statistics are stationary or quasi-stationary in form with at most non-stationary location parameters;

b) the reference and data channel first-order statistics are of identical form, differing, if at all, only in their location parameters;

c) the members of each sample pair (Y_{2i}, Y_{2i-1}), $i = 1, 2, \ldots, n/2$, have identical first-order statistics;

d) the members of each sample pair (X_{2j}, X_{2j-1}), $j = 1, 2, \ldots, m/2$, have identical first-order statistics

Proof:

Because of condition (a) we have that

$$F_{y_k}(y-M_k) = F_{o_y}(y), \text{ all } k = 1, 2, \ldots, n \tag{7.2-2}$$

and

$$F_{x_\ell}(x - N_\ell) = F_{o_x}(x) \text{ , all } \ell = 1, 2, \ldots, m \tag{7.2-3}$$

where M_k and N_ℓ are, respectively, the medians of the random variables Y_k and X_ℓ. Moreover, because of condition (b) we have that

$$F_{y_k}(y-M_k) = F_{x_\ell}(x-N_\ell) = F_o(x) \text{ all } \begin{array}{l} k = 1, 2, \ldots, n \\ \ell = 1, 2, \ldots, m \end{array} \tag{7.2-4}$$

Conditions (c) and (d) in conjunction with Eq. (7.2-4) simply state, respectively, that

$$F_{y_{2i-1}}(y-M_i) = F_{y_{2i}}(y-M_i) = F_o(y) \text{ , all } i = 1, 2, \ldots, n/2 \tag{7.2-5}$$

and

$$F_{x_{2j-1}}(x - N_j) = F_{x_{2j}}(x - N_j) = F_o(x) \text{ , all } j = 1, 2, \ldots, m/2 \tag{7.2-6}$$

Using Eq. (7.2-5) , we obtain the probability density of V_i where $V_i = Y_{2i} - Y_{2i-1}$. This is

$$f_{v_i}(v) = \int_{-\infty}^{\infty} f_{y_{2i}}(y) \; f_{y_{2i-1}}(y + v) \, dy$$

$$= \int_{-\infty}^{\infty} f_{y_{2i}}(x - M_i) \, f_{y_{2i-1}}(x + v - M_i) \, dx$$

$$= \int_{-\infty}^{\infty} f_o(x) \; f_o(x + v) \, dx \tag{7.2-7}$$

Thus, the density functions of the random variables V_i , $i = 1, 2, \ldots,$ $n/2$ are the same and given by Eq. (7.2-7) above. In the same manner and using Eq. (7.2-6), it is easily shown that the random variables U_j, $j = 1, 2, \ldots, m/2$ have the same density, given by Eq. (7.2-7) also. Thus, it has been established that the random variables V_i $i = 1, 2, \ldots, n/2$ and U_j, $j = 1, 2, \ldots, m/2$, have identical distributions. Hence, according to reference (31), the mean and variance of the modified test statistic are given by Eqs. (6.2-2) and (6.2-3) where $F_o(0)$ must, in this case, be substituted by $F_v(0)$ where $F_v(v)$ is the common distribution of V_i and U_j under no-signal conditions. Also, in this case, m and n must be substituted in Eqs. (6.2-2) and (6.2-3) by n/2 and m/2, respectively. From these expressions for the mean and variance, it is seen that a necessary and sufficient condition for the modified test statistic false-alarm rate to be asymptotically distribution-free is that zero be a specified quantile of the distribution $F_v(v)$, regardless of the channel statistics. This can be shown to be true using Eq. (7.2-7). Thus

$$F_v(0) = \int_{-\infty}^{0} f_v(v)\, dv \tag{7.2-8}$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{\infty} f_o(x)\, f_o(x + v)\, dxdv$$

$$= \int_{-\infty}^{\infty} f_o(x) \left[\int_{-\infty}^{0} f_o(x + v)\, dv \right] dx$$

$$= \int_{-\infty}^{\infty} F_o(x)\, dF_o(x)$$

$$= \frac{1}{2} \qquad \text{q.e.d.}$$

This completes the proof of the theorem.

It is seen from the above theorem that regardless of the form of the channel statistics and regardless of whether the noise medians are rapidly varying and/or unknown, the modified T-statistic remains distribution-free for the class of detection problems with reference and data channel first-order statistics of identical and stationary form, under no-signal conditions. For this class of detection problems, the mean and variance under no-signal conditions are

$$E_o\left[T_{mn}\right] = \frac{1}{4} \tag{7.2-9}$$

$$\sigma_o^2\left[T_{mn}\right] = \frac{2}{48n} \quad \text{for } m >> n >> 7 \tag{7.2-10}$$

7.2.2 The Modified T-statistic Efficacy

For the class of detection problems given by Theorem 7.1, the modified T-statistic mean under signal conditions is

$$E_\theta\left[T_{mn}\right] = \int_0^\infty \left[1-G_v(x)\right] dF_v(x) + \int_{-\infty}^0 G_v(v)\, dF_v(v) \tag{7.2-11}$$

where $G_v(x)$ and $F_v(x)$ are the distributions of the random variable V under signal and under no-signal conditions, respectively. Proceeding in the same manner as in Chapter 6, we obtain the efficacy of the modified T-statistic for $m >> n >> 7$. This is

$$K = 24\left[\int_{-\infty}^{0} \left.\frac{dG_v(x)}{d\theta}\right|_{\theta=0} dF_v(x) - \int_0^\infty \left.\frac{dG_v(x)}{d\theta}\right|_{\theta=0} dF_v(x)\right]^2 \tag{7.2-12}$$

7.2.3 Performance Indices

It was shown in the previous chapter that the T-statistic satisfies all of the conditions (A) - (F) in the weak signal case. The modified T-statistic, if expressed in terms of v_i and u_j, is equivalent to the T-statistic; hence, it too satisfies all of conditions (A) - (F) in the weak signal case and whenever $\left.\frac{dG_v(x)}{d\theta}\right|_{\theta=0}$ exists. Thus, in the weak signal case and for the class of detection problems specified by theorem 7.1, the modified T-statistic performance relation and output signal-to-noise ratio are given by

$$24\left[\int_{-\infty}^{0} \frac{dG_v(x)}{d\theta}\bigg|_{\theta=0} dF_v(x) - \int_{0}^{\infty} \frac{dG_v(x)}{d\theta}\bigg|_{\theta=0} dF_v(x)\right]^2 \bar{\theta}^2 n =$$

$$= 2\left[\operatorname{erf}^{-1}(1-2\alpha) + \operatorname{erf}^{-1}(1-2\beta)\right]^2$$

(7.2-13)

and

$$\left(\frac{S}{N}\right) = \bar{\theta}\sqrt{24n}\left[\int_{-\infty}^{0} \frac{dG_v(x)}{d\theta}\bigg|_{\theta=0} dF_v(x) - \int_{0}^{\infty} \frac{dG_v(x)}{d\theta}\bigg|_{\theta=0} dF_v(x)\right]$$

(7.2-14)

The efficacy given in Eq. (7.2-12) may also be used to obtain the asymptotic relative efficiency of T_{mn} with respect to other statistics. Thus, the efficacy specifies all the performance indices of the adaptive T-detector for the conditions under which the efficacy given in Eq. (7.2-12) is valid.

7.3 Applications

In the following, the adaptive T-detector is applied to the detection of a gaussian signal in gaussian noise; its performance in the problem is evaluated and compared to the performance in the same problem of the T-detector and of the optimum likelihood detector.

7.3.1 Detection of Narrow-Band White Gaussian Signal in Additive White Gaussian Noise

The probability density functions $g_v(x)$ and $f_v(x)$ are, in this case, given by

$$f_v(x) = \frac{1}{\sqrt{2\pi 2}}\, e^{-\frac{x^2}{2\cdot 2}}, \quad -\infty < x < \infty \tag{7.3-1}$$

$$g_v(x) = \frac{1}{\sqrt{2\pi 2(\theta+1)}} e^{-\frac{x^2}{2\cdot 2(\theta+1)}}, \quad -\infty < x < \infty \tag{7.3-2}$$

Thus,

$$\left.\frac{dG_v(x)}{d\theta}\right|_{\theta=0} = -\frac{x}{2} \quad f_v(x) \tag{7.3-3}$$

The adaptive detector efficacy is obtained using Eqs. (7.3-1) and (7.3-3) in Eq. (7.2-12). The efficacy is

$$K = 24 \left[\int_0^\infty \frac{x}{8\pi} e^{-\frac{x^2}{2}} dx - \int_{-\infty}^0 \frac{x}{8\pi} e^{-\frac{x^2}{2}} dx \right]^2 \tag{7.3-4}$$

$$= \frac{1}{2} \frac{3}{\pi^2}$$

Using Eqs. (6.3-1) and (7.3-4) we obtain the asymptotic relative efficiency of the adaptive T-detector with respect to the T-detector. The ARE is

$$ARE_{T_{mn},\ T_{mn}} (M,N) = \frac{1}{2} \tag{7.3-5}$$

The asymptotic relative efficiency of the adaptive T-detector with respect to the optimum likelihood detector is obtained using Eqs. (B-59) and (7.3-4). The ARE is

$$ARE_{T_{mn},\ L_n^*} = 0.305 \tag{7.3-6}$$

The results for this problem are presented graphically in Fig. 18.

7.4 Summary of Results

In this chapter, a modified version of the T-detector that is adaptive to rapid changes in the location parameters — specifically

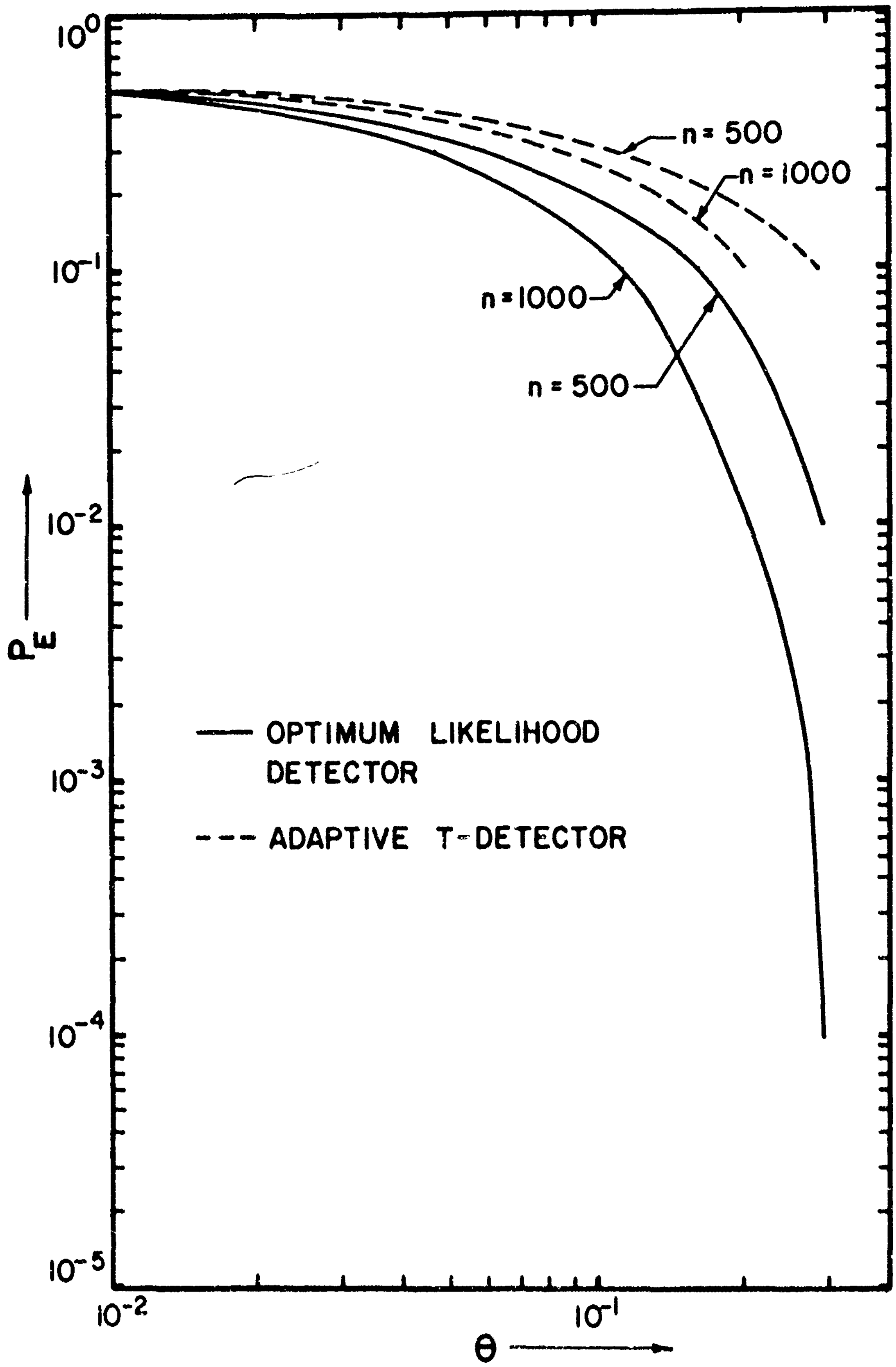

Fig. 18. Probability of Error vs. Received Mean SNR in the Detection of a Gaussian Signal in Gaussian Noise.

the medians and/or to unknown medians - was proposed and investigated. The conditions under which the adaptive T-detector remains distribution-free were also obtained. It was shown that the adaptive T-detector false alarm rate remains asymptotically distribution-free for the class of detection problems with reference and data channel first-order statistics that are of identical and stationary form, under no-signal conditions.

The adaptive T-detector was applied to the detection of a gaussian signal in gaussian noise, and its performance investigated. It was found that the adaptive T-detector is half as efficient as the T-detector. This, however, is expected since the adaptive detector utilizes for detection only half as many samples as the T-detector; the other half is used to make the adaptive detector distribution-free for a wide class of problems even when the medians are rapidly varying and/or unknown. The adaptive T-detector information rate was found to be 30% of that of the optimum likelihood detector.

From the results obtained in this chapter, it is concluded that use of the adaptive T-detector instead of a likelihood detector entails a small loss of detection efficiency for gaussian channel statistics. However, the adaptive T-detector is applicable even when the form of the distributions is unknown, since it remains distribution-free for a wide class of detection problems.

Chapter 8

CONCLUSION

8.1 Summary of Problem Discussion and Procedures

We have been concerned in this work with a class of two-input detection systems for digital communication over random and unknown channels. The two-input systems herein investigated possess false-alarm rates that are invariant for wide classes of channel statistics. The motivation for considering such systems arises from the need of insuring an acceptable performance in a changing and/or incompletely known environment.

Specifically, in this work, coincidence detection procedures with invariant or distribution-free false-alarm rates were proposed and investigated. In the distribution-free coincidence procedures investigated, the threshold was chosen to be a specified noise distribution quantile (i.e., the median) so that the test statistic possessed, asymptotically and under no-signal conditions, a known distribution, independent of the statistics of the detection problem.

The coincidence detection procedures were subsequently modified so that the detectors based on them constituted learning systems with respect to slowly varying and/or known location parameters. The coincidence procedures were modified in still another manner so that the detectors utilizing these modified procedures constituted adaptive systems with respect to rapidly varying and/or unknown location parameters.

The distribution-free coincidence detectors were applied to various detection problems of practical importance; their performances were evaluated and compared to the performance of comparable likelihood detectors.

In addition to the distribution-free coincidence detectors, a detector well suited for the detection of stochastic signals in noise was proposed and investigated. The T-statistic was subsequently modified so that the detector utilizing the modified statistic constituted an adaptive system with respect to rapidly varying and/or unknown location parameters. After obtaining the wide classes of detection problems for which the T-detector and the adaptive T-detector false-alarm rates remained distribution-free, the detectors were then applied to detection problems of practical importance; their performances were evaluated and compared to that of the optimum likelihood detector.

8.2 Conclusions

The invariant nature of the test statistic distribution under no-signal conditions insured a false-alarm invariant with respect to changes in the channel statistics. The median of the noise under no-signal conditions was the only information concerning the channel statistics that was required by the distribution-free coincidence procedures.

Also obtained were the classes of detection problems for which the false-alarm rates of the above coincidence procedures remained distribution-free.

It was found that distribution-free coincidence detectors were quite efficient, though sub-optimal, for channels with gaussian statistics, and highly efficient for channels having a combination of gaussian and impulse noise.

The T-detector and the adaptive T-detector were found to be reasonably efficient for the detection of gaussian signals in gaussian noise and highly efficient for some non-gaussian channel statistics.

In general, from the results obtained in this investigation, it is concluded that use of the distribution-free detectors proposed here, instead of equivalent likelihood detectors, entails only a small loss of detection efficiency for gaussian channel statistics; while in the case of impulse and gaussian noise present in the channel, use of the distribution-free detectors results in higher detection efficiency. Moreover, the distribution-free detectors have invariant false-alarm rates for wide classes of channel statistics - hence, they are applicable even when the form of the probability distributions is unknown. In addition, the detectors proposed herein have invariant and simple structures and can, therefore, be easily implemented.

8.3 Recommendation for Further Study

The distribution-free detectors proposed in this investigation merit further consideration. In particular, the performance of these detectors for large signal-to-noise ratios needs to be investigated. It would also be of interest to investigate their performance and distribution-free nature for the case of dependent samples. The above studies are in general difficult to do theoretically; hence, a computer simulation study and/or experimental investigation could be substituted.

The distribution-free coincidence detection procedures investigated utilized the median under no-signal conditions as their threshold level. However, other distribution quantiles could also be used. An investigation of the properties and evaluation of the performance, in detection problems of practical importance, of coincidence detection procedures utilizing as threshold levels quantiles other than the median, would constitute an important extension of the present work.

Finally it would be of interest to investigate the distribution-free nature and detection efficiency of a generalized coincidence detection procedure employing as test statistic a weighted sum of coincidence type test statistics having various distribution quantiles as their respective threshold levels. In connection with the generalized coincidence procedures, it would be of importance to obtain a weighting procedure that minimizes the variance of the generalized coincidence test statistic or better yet to obtain a weighting procedure that maximizes the information rate.

REFERENCES

REFERENCES

1. Pearson, E. S., "The Analysis of Variances in Cases of Non-normal Variation," Biometrika, Vol. 23, pp. 114-134, 1931

2. Cayen, A. A., "The Distribution of the Variance Ratio in Random Samples of Any Size Drawn from Non-normal Universes," Biometrika, Vol. 37, pp. 236-255, 1950.

3. Middleton, D., An Introduction to Statistical Communication Theory, McGraw-Hill Book Co., Inc., New York, 1960.

4. Capon J., "Nonparametric Methods for the Detection of Signals in Noise," Department of Electrical Engineering Technical Report No. T-1/N, Columbia University, March 12, 1959.

5. Capon J., "A Nonparametric Technique for the Detection of a Constant Signal in Additive Noise," IRE WESCON Convention Record, Part 4, pp. 92-103, 1959.

6. Capon J., "Locally Optimum Nonparametric Methods for the Detection of Signals in Noise," IEEE. Second Symposium on Adaptive Processes, pp. 64-77, September, 1963.

7. Groginsky, H. L., "Adaptive Detection of Statistical Signals in Noise," U.S. Navy Journal of Underwater Acoustics, Vol. 11, No. 2, April, 1961.

8. Rushforth, C. K. and Daly, R. F., "Nonparametric Detection of a Signal of Known Form in Additive Noise," Symposium on Signal Statistics, University of Washington, December, 1963.

9. Hancock, J. C. and Lainiotis, D. G., "Advanced Communication Theory Techniques," Purdue University, School of Electrical Engineering, Technical Report No. EE 63-2, March, 1963.

10. Wolff, S. S. et al., "The Polarity Coincidence Correlator: A non-parametric Detection Device," IRE Transactions on Information Theory, Vol. IT-8, No. 1, pp. 5-9, January, 1962.

11. Kanefsky, M. and Thomas, J. B., "On Adaptive Detectors for Two Input Systems," IEEE Second Symposium on Adaptive Processes, pp. 64-77, September, 1963.

12. Eykhoff, P. "Adaptive and Optimalizing Control Systems," IRE Transactions on Automatic Control, Vol. AC - 5, No. 2, pp. 148-151, June, 1960.

13. Zadeh, L. A., "On the Definition of Adaptivity," Proceedings of the IEEE, Vol. 51, No. 3, pp. 469-470, March 1963.

14. Harrington, J. V., "Analysis of the Detection of Repeated Signals in Noise by Binary Integration," IRE Transactions on Information Theory, Vol. IT-1, p. 1, January 1955.

15. Schwartz, M., "A Coincidence Procedure for Signal Detection," IRE Transactions on Information Theory, Vol. IT-2, p. 135, April 1956.

16. Capon, J., "Optimum Coincidence Procedures for Detecting Weak Signals in Noise," IRE International Convention Record, Part 4, pp. 154-166 1960.

17. Bunimovich, V. I., and Morozov, V. A., "Small Signal Reception by the Method of Binary Integration" Radio Engineering and Electronic Physics, No. 11, pp. 1734-1740, November 1962.

18. Hoeffding, W., "A Non-parametric Test of Independence," Annals of Math. Statistics, Vol. 19, p. 546, 1948.

19. Lehmann, E. L., "Consistency and Unbiassedness of Certain Non-parametric Tests," Annals of Math. Statistics, Vol. 22, p. 165, 1951.

20. Pitman, E. J. G., "Lecture Notes on Non-parametric Statistical Inference," Columbia University, Spring 1948.

21. Noether, G. E., "On a Theorem of Pitman," Annals Math. Statistics, Vol. 26, p. 64, 1955.

22. Faser, D. A. S., Nonparametric Methods in Statistics, John Wiley and Sons, Inc., New York, 1957.

23. Rudin, W., Principles of Mathematical Analysis, Mc Graw-Hill Book Co., Inc., New York, 1953.

24. Shepelavey, B., "Non-gaussian Atmospheric Noise in Binary-Data Phase Coherent Communication Systems," IEEE Transactions on Communication Systems, Vol. CS-11, No. 3, September 1963.

25. Rice, S. O., "Statistical Properties of a Sine Wave Plus Random Noise," Bell System Technical Journal, Vol. 27, p. 109, 1948.

26. Rice, S. O., "Mathematical Analysis of Random Noise," p. 238 in Noise and Stochastic Processes, ed. N. Wax, Dover Publications, Inc., New York, 1954.

27. Davenport, W. B., and Root, W. L., An Introduction to the Theory of Random Signals and Noise, Mc Graw-Hill Book Co., Inc., New York, 1958.

28. Morrow, W. E., "Channel Characterization: Basic Approach," Ch. 4 in Lectures on Communication System Theory, ed. E. J. Baghdady, Mc Graw-Hill Book Co., Inc., New York, 1961.

29. Hodges, J. L., and Lehman, E. L., "The Efficiency of Some Non-parametric Competitors of the T-test," Annals of Math. Statistics, Vol. 27, p. 324, 1956.

30. Cramer, H., Mathematical Methods of Statistics, Princeton University Press, Princeton, N.J., 1946.

31. Sukhatme, B. V., "On Certain Two Sample Nonparametric Tests for Comparing Variances," Annals of Math. Statistics, Vol. 28, p. 189-194, 1957.

APPENDIX

Appendix A

DETECTION PROBLEMS INVESTIGATED

A-1 Detection of a Constant in Additive Noise - General Case

In this problem, the signal is either one of constant amplitude or a sine wave of known phase sampled always at the same point, preferably at its peak, so that as far as the samples are concerned, this is equivalent to a signal of constant amplitude. The signal-to-noise ratio θ is defined as

$$\theta = \frac{A}{\sigma} \tag{A-1}$$

where σ^2 is the noise variance and A is the amplitude of the constant signal in the case of a constant signal, or the peak amplitude of the sine wave in the case of a sinusoidal signal.

The probability distribution function $F_o(y)$ under no-signal conditions and the probability distribution $G_\theta(y)$ under signal conditions are related as follows

$$G_\theta(y) = F_o(y - \theta) \tag{A-2}$$

hence (4)

$$\left.\frac{dG_\theta(y)}{d\theta}\right|_{\theta=0} = \left.\frac{dF_o(y - \theta)}{d\theta}\right|_{\theta=0} = - f_o(y) \tag{A-3}$$

where $f_o(y)$ is the probability density function under no-signal conditions.

A-2 Detection of a Constant in Additive Gaussian Noise

This detection problem is a specific case of the previous general problem, with

$$f_o(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad -\infty < y < \infty \tag{A-4}$$

Hence

$$\left.\frac{dG_\theta(y)}{d\theta}\right|_{\theta=0} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \quad -\infty < y < \infty \tag{A-5}$$

A-3 Detection of a Constant in Additive Combination of Gaussian and Impulse Noise

This detection problem is also a specific case of the general problem treated in section (A-1), with probability density under no-signal conditions $f_o(y)$ given by

$$f_o(y) = a\ e^{-b|y|^c}, \quad -\infty < y < \infty \tag{A-6}$$

This form of noise was chosen because it realistically represents (11, 24) the amplitude statistics of a noise source consisting of an additive combination of gaussian and impulse noise. The relationship between the parameters a, b, and c can be derived from the following equations

$$\int_{-\infty}^{\infty} f_o(y)\ dy = 1$$

$$\int_{-\infty}^{\infty} y^2 f_o(y)\ dy = 1$$

the latter equation simply ensuring a noise variance of one. The relation of interest is

$$4a^2 = c^2 \frac{\Gamma\left(3/c\right)}{\Gamma^3\left(1/c\right)} \tag{A-7}$$

For this problem

$$\left.\frac{dG_\theta(y)}{d\theta}\right|_{\theta=0} = -\,a\,e^{-b|y|^c}\,, \quad -\infty < y < \infty \tag{A-8}$$

A-4 Detection of a Sine Wave of Unknown Phase in Additive Gaussian Noise

The distribution function under signal conditions for this detection problem has been shown to be (25)

$$G_\theta(y) = \frac{1}{\pi}\int_0^\pi \Phi\left(\frac{y}{\sigma} - 2\sqrt{\theta}\cos y\right) dy \tag{A-9}$$

$$= \Phi\left(\frac{y}{\sigma}\right) + \sum_{K=1}^{\infty} \frac{\sigma^{2K-1}\theta^K}{K!\,K!} \frac{d^{(2K-1)}}{dy^{(2K-1)}} \phi\left(\frac{y}{\sigma}\right)$$

where the mean of the noise is assumed to be zero, σ^2 is the mean square value of the noise, A is the maximum amplitude of the sine wave and θ is defined as

$$\theta = \frac{A^2}{2\sigma^2} \tag{A-10}$$

that is, θ is the ratio of the mean square value of the signal to the mean square value of the noise. The functions $\phi(y)$ and $\Phi(y)$ are defined as

$$\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad -\infty < y < \infty \tag{A-11}$$

$$\Phi(y) = \int_{-\infty}^{y} \phi(x)\, dx, \quad -\infty < y < \infty \tag{A-12}$$

The distribution $F_o(y)$ under no-signal conditions is found from Eq. (A-9) by setting $\theta = 0$

$$F_o(y) = \Phi\left(\frac{y}{\sigma}\right) \tag{A-13}$$

Differentiating Eq. (A-9) gives

$$\left.\frac{dG_\theta(y)}{d\theta}\right|_{\theta=0} = -\frac{y}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}, \quad -\infty < y < \infty \tag{A-14}$$

A-5 <u>Envelope Detection of a Sine Wave in Narrow-Band Gaussian Noise</u>

In this detection problem, the observed waveform is the envelope of a sine wave and additive narrow-band noise. The frequency of the sine wave is the same as the center frequency of the noise baud. The noise is assumed to be a gaussian random process with zero mean. Under these conditions, the distribution functions under signal and under no-signal conditions are (26)

$$g_\theta(y) = \frac{y}{\sigma^2} e^{-\left[\frac{y^2}{2\sigma^2} + \theta\right]} I_o\left[\frac{y}{\sigma}(2\theta)^{1/2}\right], \quad y \geq 0 \tag{A-15}$$

$$= 0, \quad y < 0$$

$$f_o(y) = \frac{y}{\sigma^2} e^{-\left[\frac{y^2}{2\sigma^2}\right]}, \quad y \geq 0 \tag{A-16}$$

$$= 0, \quad y < 0$$

where y is the amplitude of the envelope, σ^2 is the mean square value of the noise, $I_o(x)$ is the modified Bessel function of the first kind, zero-th order, and the signal-to-noise ratio θ is equal to the signal-to-noise power ratio; namely

$$\theta = \frac{A^2}{2\sigma^2} \qquad \text{(A-17)}$$

where A is the peak amplitude of the sine wave.

The distribution function under no-signal conditions is

$$\begin{aligned} F_o(y) &= 1 - e^{-\frac{y^2}{2\sigma^2}} , && y \geq 0 \qquad \text{(A-18)} \\ &= 0 , && y < 0 \end{aligned}$$

From Eq. (A-15) we obtain the distribution function under signal conditions, which in turn gives (4)

$$\begin{aligned} \left.\frac{dG_\theta(y)}{d\theta}\right|_{\theta=0} &= -\frac{y^2}{2\sigma^2} e^{-\frac{y^2}{2\sigma^2}} , && y \geq 0 \qquad \text{(A-19)} \\ &= 0 , && y < 0 \end{aligned}$$

A-6 Square-Law Detection of a Sine Wave in Narrow-Band Gaussian Noise

In this detection problem the signal is again a sine wave immersed in additive narrow-band noise. The frequency of the sine wave is the same as the center frequency of the noise band. The noise is assumed to be a gaussian random process with mean zero and mean square value one. The observed waveform is the output of a square-law detector. The probability densities of the output of the square-law detector under signal and under no-signal conditions are

$$g_{\theta}(y) = e^{-(y+\theta)} \; I_o[2\sqrt{\theta y}] \,, \qquad y \geq 0 \tag{A-20}$$

$$= 0 \,, \qquad y < 0$$

$$f_o(y) = e^{-y} \,, \qquad y \geq 0 \tag{A-21}$$

$$= 0 \,, \qquad y < 0$$

where $I_o(x)$ is the modified Bessel function defined previously, and the signal-to-noise ratio θ is equal to the signal-to-noise power ratio; namely

$$\theta = \frac{A^2}{2\sigma^2} \tag{A-22}$$

where A is the peak amplitude of the sine wave, and σ^2 is the mean square value of the noise in this case equal to one. From Eq. (A-20) we obtain

$$\left. \frac{dG_{\theta}(y)}{d\theta} \right|_{\theta=0} = - y \, e^{-y} \,, \qquad y \geq 0 \tag{A-23}$$

$$= 0 \,, \qquad y < 0$$

A-7 <u>Detection of Narrow-Band White Gaussian Signal in Additive Narrow-Band White Gaussian Noise</u>

In this problem, the observed waveform is a sample function from a random process which is the sum of a narrow-band white gaussian signal process and a narrow-band white gaussian noise process. The processes are centered at the same frequency and have zero means. The probability densities of the detector input under signal and under no-signal conditions, respectively, are

$$g_\theta(y) = \frac{1}{\sigma_N \sqrt[2]{2\pi(1+\theta)}} e^{-\frac{y^2}{2\sigma_N^2(1+\theta)}}, \qquad -\infty < y < \infty \qquad \text{(A-24)}$$

$$f_o(y) = \frac{1}{\sigma_N \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_N^2}}, \qquad -\infty < y < \infty \qquad \text{(A-25)}$$

where σ_N^2 is the mean square value of the noise, and the signal-to-noise ratio θ is equal to the signal-to-noise power ratio

$$\theta = \frac{\sigma_S^2}{\sigma_N^2} \qquad \text{(A-26)}$$

where σ_S^2 is the mean square vlaue of the signal.

The distribution function of the detector input under signal conditions, obtained from Eq. (A-24), is given by

$$G_\theta(y) = \int_{-\infty}^{\frac{y}{\sqrt{1+\theta}}} \frac{1}{\sigma_N \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_N^2}} dy \qquad \text{(A-27)}$$

$$= F_o\left(\frac{y}{\sqrt{1+\theta}}\right)$$

Differentiating, we obtain

$$\left.\frac{dG_\theta(y)}{d\theta}\right|_{\theta=0} = -\frac{y}{2} f_o(y) \qquad \text{(A-28)}$$

A-8 Envelope Detection of Narrow-Band White Gaussian Signal in Additive Narrow-Band White Gaussian Noise

In this problem, the signal is again a narrow-band white gaussian random process immersed in an additive narrow-band white gaussian noise process. The signal and noise random processes are assumed to be centered at the same frequency and to have zero means. However, in this problem, the input waveform prior to its examination by the detector is passed through a linear envelope detector. Thus, the observed waveform is the envelope of a sample function from a random process that is the sum of two narrow-band white gaussian processes. The probability densities of the envelope under signal and under no-signal conditions are (27)

$$g_\theta(y) = \frac{y}{\sigma_N^{\,2}(1+\theta)}\, e^{-\frac{y^2}{2\sigma_N^{\,2}(1+\theta)}}, \quad y \geq 0 \qquad \text{(A-29)}$$

$$= 0, \qquad y < 0$$

$$f_o(y) = \frac{y}{\sigma_N^{\,2}}\, e^{-\frac{y^2}{2\sigma_N^{\,2}}}, \quad y \geq 0 \qquad \text{(A-30)}$$

$$= 0, \qquad y < 0$$

where $\sigma_S^{\,2}$ and $\sigma_N^{\,2}$ are, respectively, the signal and noise mean square values. The signal-to-noise ratio θ is equal to the signal-to-noise power ratio

$$\theta = \frac{\sigma_S^{\,2}}{\sigma_N^{\,2}} \qquad \text{(A-31)}$$

The distribution function $G_\theta(y)$ under signal conditions is

$$G_\theta(y) = \int_0^{\frac{y}{\sqrt{1+\theta}}} \frac{y}{\sigma_N^2} e^{-\frac{y^2}{2\sigma_N^2}} dy \qquad \text{(A-32)}$$

$$= F_o\left(\frac{y}{\sqrt{1+\theta}}\right)$$

Differentiating, we obtain

$$\left.\frac{dG_\theta(y)}{d\theta}\right|_{\theta=0} = -\frac{y}{2} f_o(y) \qquad \text{(A-33)}$$

A-9 <u>Square-Law Detection of Narrow-Band White Gaussian Signal in Additive Narrow-Band White Gaussian Noise</u>

In this detection problem, as in the previous two, in the absence of signal, the channel output is a sample function of the noise narrow-band white gaussian process; and, in the presence of signal, the channel output is the sum of two sample functions — one from the noise and the other from the narrow-band white gaussian signal process. The signal and noise processes are assumed to be centered at the same frequency and to have zero means. The channel output prior to its examination by the detector is passed through a square-law detector (27). The probability densities of the square-law detector output under signal and under no-signal conditions, respectively, are (27)

$$g_\theta(y) = \frac{1}{\sigma_N^2(1+\theta)} e^{-\frac{y}{\sigma_N^2(1+\theta)}} , \quad y \geq 0 \qquad \text{(A-34)}$$

$$= 0 , \qquad y < 0$$

$$f_o(y) = \frac{1}{\sigma_N^2} e^{-\frac{y}{\sigma_N^2}} , \qquad y \geq 0 \qquad \text{(A-35)}$$

$$= 0 , \qquad y < 0$$

where σ_S^2 and σ_N^2 are, respectively, the signal and noise mean square values and θ is the signal-to-noise power ratio

$$\theta = \frac{\sigma_S^2}{\sigma_N^2} \qquad \text{(A-36)}$$

The distribution function under signal conditions is

$$G_\theta(y) = \int_0^{\frac{1}{1+\theta}} f_o(x)\, dx \qquad \text{(A-37)}$$

$$= F_o\left(\frac{y}{1 + \theta}\right)$$

Differentiating, we obtain

$$\left.\frac{dG_\theta(y)}{d\theta}\right|_{\theta=0} = - y\, f_o(y) \qquad \text{(A-38)}$$

Appendix B

LIKELIHOOD DETECTORS

In this appendix, the likelihood detectors associated with the detection problems treated in this investigation are presented, and their performance in the above problems evaluated.

It is well known that a likelihood detector bases its decisions on the likelihood ratio statistic defined as

$$L_n = \prod_{i=1}^{n} \frac{g_\theta(y_i)}{f_o(y_i)} \tag{B-1}$$

where n is the number of independent samples extracted from the observed waveform Y(t), and $g_\theta(y)$, $f_o(y)$ are the probability densities of the detector input under signal and under no-signal conditions, respectively.

For the weak signal case, and provided the derivative of $g_\theta(y)$ with respect to θ exists and is continuous at $\theta = 0$, the likelihood ratio statistic is equivalent to

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} \frac{b'(y_i)}{f_o(y_i)} \tag{B-2}$$

where

$$b'(y) = \left. \frac{d\, g_\theta(y)}{d\theta} \right|_{\theta = 0} \tag{B-3}$$

The likelihood statistic given in Eq. (B-2) satisfies condition (A)-(F) in the weak signal case and for the problems investigated. Thus, its performance relation and output signal-to-noise ratio are given by

$$K^{*}\theta^{2}n = 2[\text{erf}^{-1}(1-2\alpha) + \text{erf}^{-1}(1-2\beta)]^{2} \tag{B-4}$$

$$\left(\frac{S}{N}\right) = \theta\sqrt{nK^{*}} \tag{B-5}$$

where the efficacy K^* is (5)

$$K = \int \frac{b'^{2}(y)}{f_{o}(y)}\, dy \tag{B-6}$$

The specific likelihood detectors associated with each particular detection problem and their efficacy in the problem are given below.

B-1 Detection of a Constant in Additive Gaussian Noise

Case I. The additive noise statistics are assumed to be stationary. Thus, the mean and variance under no-signal conditions of the detector input will be assumed to be known, since they can easily be obtained for a stationary process. The mean under no-signal conditions can then be subtracted from the reference and data waveforms, and the resulting waveforms divided by the variance so the random variable Y representing the amplitude of the detector input is normalized to a N(0, 1) random variable. Eqs. (A-2) and (A-4) are then applicable. Utilizing them in Eqs. (B-2) and (B-3) we obtain (5)

$$L_{n}^{*} = \frac{1}{n}\sum_{i=1}^{n} y_{i} \tag{B-7}$$

$$K^{*} = 1 \tag{B-8}$$

So, for this case of known mean and variance under no-signal conditions, only a data sample is required by the likelihood detector.

<u>Case II</u>. In this case the mean under no-signal conditions is assumed to be unknown or quasi-stationary, while the variance under no-signal conditions is assumed to be stationary so that, if unknown, it can be easily obtained. Thus, the input can be normalized to have a variance of one. For this case the probability densities of the detector input are

$$f_o(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-M)^2}{2}}, \qquad -\infty < y < \infty \tag{B-9}$$

$$g_\theta(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-M-\theta)^2}{2}}, \qquad -\infty < y < \infty \tag{B-10}$$

where

$$\begin{aligned} \theta &= \frac{A}{\sigma} \\ &= A \end{aligned} \tag{B-11}$$

Thus

$$\begin{aligned} b'(y) &= \frac{(y-M)}{\sqrt{2\pi}} e^{-\frac{(y-M)^2}{2}}, \qquad -\infty < y < \infty \\ &= (y-M)\, f_o(y) \end{aligned} \tag{B-12}$$

and the likelihood ratio statistic is

$$L_n^*(M) = \frac{1}{n} \sum_{i=1}^{n} (y_i - M) \tag{B-13}$$

However, the mean M is unknown. To apply the above statistic, the unknown mean will be estimated from a reference sample function obtained under no-signal conditions. The sample mean $\overline{M}$ is chosen as the estimator of M.

$$M = \frac{1}{m} \sum_{j=1}^{m} x_j \tag{B-14}$$

where x_j is the value of the j^{th} sample obtained from $N'(t)$, the reference sample function. Utilizing $\overline{M}$, the likelihood statistic becomes

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} (y_i - \overline{M}) \tag{B-15}$$

$$= \frac{1}{n} \sum_{i=1}^{n} y_i - \frac{1}{m} \sum_{j=1}^{m} x_j$$

The mean of L_n^* under signal conditions is

$$E_\theta[L_n^*] = \theta \tag{B-16}$$

and the variance under no-signal conditions is given by

$$\sigma_o^2[L_n^*] = \frac{1}{n} + \frac{1}{m} \tag{B-17}$$

Hence, the efficacy of the likelihood statistic is

$$K^* = \frac{1}{n} \left[\frac{\left. \frac{d\, E_\theta(L_n^*)}{d\theta} \right|_{\theta=0}}{\sigma_o(L_n^*)} \right]^2 \tag{B-18}$$

$$= \frac{1}{1 + \frac{n}{m}}$$

<u>Case III</u>. The mean under no-signal conditions is assumed to be non-stationary, while the variance is assumed to be stationary. This is a conceivable practical situation. Since the mean is non-stationary and its time variation is unknown, it cannot be estimated; hence, the likelihood detector cannot be employed if only one sample function is used. However, if two channels -- a reference and a data channel with identical statistics -- are utilized, we may eliminate the need for knowing the mean if the reference sample function is subtracted from the data sample function and a decision is based on the samples extracted from the difference waveform. If, in addition, the difference waveform is divided by the known variance, then the probability densities of the amplitude of the detector input Z are independent of the time varying mean, and are given by

$$f_o(z) = \frac{1}{\sqrt{4\pi}} e^{-\frac{z^2}{4}}, \qquad -\infty < z < \infty \tag{B-19}$$

$$g_\theta(z) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(z-\theta)^2}{2}}, \qquad -\infty < z < \infty \tag{B-20}$$

Thus

$$b'(z) = \frac{1}{2} z \; f_o(z)$$

and

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} \frac{z_i}{2} \tag{B-21}$$

$$K^* = \frac{1}{2} \tag{B-22}$$

B-2 Detection of a Constant in Additive Noise-Unspecified Distributions

In this general problem the form of the distributions $G_\theta(y)$ and $F_o(y)$ are unknown. The only information available is that $G_\theta(y)$ and $F_o(y)$ obey the relation

$$G_\theta(y) = F_o(y - \theta) \tag{B-23}$$

Thus, the likelihood ratio statistic cannot be used in this case of distributions of unknown form. However, the likelihood ratio statistic obtained under a similar gaussian situation can be, and usually is, employed. This is given by Eq. (B-7).

Case I. The mean and variance under no-signal conditions are assumed to be stationary; hence, they can be easily obtained if unknown and the detector input amplitude Y normalized, so that

$$E_o[Y] = 0 \tag{B-24}$$

$$\sigma_o^2[Y] = 1$$

The likelihood statistic utilized in this case is

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} y_i$$

with efficacy given (5) by

$$K^* = 1 \tag{B-25}$$

Case II. The mean is unknown or quasi-stationary under no-signal conditions, while the variance is assumed stationary and known. The detector utilizes the statistic

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} y_i - \frac{1}{m} \sum_{j=1}^{m} x_j \tag{B-26}$$

Since for this general problem $g_\theta(y) = f_o(y-\theta)$ we have

$$\begin{aligned} E_\theta[L_n^*] &= \frac{1}{n} \sum_{i=1}^{n} E[Y_i] - \frac{1}{m} \sum_{j=1}^{m} E[X_j] \qquad \text{(B-27)} \\ &= \theta + M - M \\ &= \theta \end{aligned}$$

and

$$\sigma_o^{\,2}[L_n^*] = \frac{1}{n} + \frac{1}{m} \tag{B-28}$$

for an input normalized with respect to variance. Hence, the efficacy is

$$K^* = \frac{1}{1 + \frac{n}{m}} \tag{B-29}$$

Case III. The mean is non-stationary under no-signal conditions, and its time variation is unknown. The variance is assumed to be stationary and known. The mean cannot be estimated under the stated conditions. Hence, the likelihood detector, in order not to depend on the mean, will base its decision on measurements made on the difference between data and reference sample functions obtained from channels of independent but identical statistics. The difference waveform is also normalized so as to have variance of one. Under the stated conditions

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} z_i \tag{B-30}$$

where

$$Z(t) = Y(t) - N\ (t)$$

Since $G_\theta(y) = F_o(y-\theta)$, we have that

$$E_\theta(Z) = \theta$$

and (B-31)

$$\sigma_o^{\ 2}(Z) = 2$$

Thus

$$E_\theta[L_n^*] = \theta$$

(B-32)

$$\sigma_o^{\ 2}[L_n^*] = \frac{2}{n}$$

Hence the efficacy is

$$K^* = \frac{1}{2} \tag{B-33}$$

B-3 Detection of a Constant in Additive Combination of Gaussian and Impulse Noise

This detection problem is a specific case of the general problem discussed in the previous section. Hence, as stated there, if the forms of the distributions are not known, then the binary integrator of

Eq. (B-7) will be used since, in the absence of knowledge concerning the form of the distributions, the likelihood ratio statistic obtained under the gaussian assumption is usually employed. The efficacies of the statistic for the present detection problem are those given by Eqs. (B-25), (B-29) and (B-33).

B-4 Detection of a Sine Wave of Unknown Phase in Additive Gaussian Noise

Case I. The mean and variance under no-signal conditions are assumed stationary and known. For this case the likelihood ratio statistic and its efficacy are given (16) by

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{y_i^2}{\sigma^2} - 1 \right) \tag{B-34}$$

and

$$K^* = 2 \tag{B-35}$$

Case II. The mean under no-signal conditions is assumed to be non-stationary, while the variance is assumed stationary and known. In this case, as stated previously, the likelihood detector will base its decisions on measurements made on Z(t), the difference between the data and reference sample functions normalized to have unit variance. Thus, the amplitude Z of the detector input is given by

$$Z = Y - X \tag{B-36}$$

where Y and X are the amplitudes of the data and reference channel outputs. Since Y and X are assumed to be independent, the probability densities of Z under signal and under no-signal conditions are

$$g_\theta(z) = \int_{-\infty}^{\infty} g_\theta(y)\, f_o(y-z)\, dy \tag{B-37}$$

$$f_o(z) = \int_{-\infty}^{\infty} f_o(y)\, f_o(y-z)\, dy \tag{B-38}$$

Utilizing Eqs. (A-9) and (A-13) we obtain

$$b'(z) = \int_{-\infty}^{\infty} (y^2 - 1)\, f_o(y)\, f_o(y-z)\, dy \tag{B-39}$$

$$= \frac{1}{2}\left(\frac{z^2}{2} - 1\right) f_o(z), \qquad -\infty < z < \infty$$

Thus, the likelihood ratio statistic and its efficacy are

$$L_n^* = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{z_i^2}{4} - \frac{1}{2}\right) \tag{B-40}$$

and

$$K^* = \frac{1}{2} \tag{B-41}$$

B-5 <u>Envelope Detection of a Sine Wave in Narrow-Band Gaussian Noise</u>

<u>Case I</u>. The assumptions for this case are the same as those made in Case I of Section B-4. The likelihood ratio statistic and its efficacy are given (5) by

$$L_n^* = \frac{1}{n}\sum_{i=1}^{n} (y_i^2 - 1) \tag{B-42}$$

$$K^* = 1 \tag{B-43}$$

Case II. The assumptions are the same as those made in Case II of the previous section. Hence, for the difference envelope random variable V, utilizing Eqs. (A-15) and (A-16) we have

$$
\begin{aligned}
g_\theta(v) &= \frac{v}{2}\, e^{-\left[\frac{v^2}{4} + \frac{\theta}{2}\right]}\, I_o\left(v\sqrt{\frac{\theta}{2}}\right), && v > o \\
&= o\,, && v < o
\end{aligned}
\qquad \text{(B-44)}
$$

$$
\begin{aligned}
f_o(v) &= \frac{v}{2}\, e^{-\frac{v^2}{4}}, && v \geq o \\
&= o\,, && v < o
\end{aligned}
\qquad \text{(B-45)}
$$

Hence

$$
b'(v) = \frac{1}{2}\left(\frac{v^2}{4} - 1\right) f_o(v) \qquad \text{(B-46)}
$$

and the likelihood ratio statistic and its efficacy are

$$
L_n^* = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2}\left(\frac{v_i^2}{4} - 1\right) \qquad \text{(B-47)}
$$

$$
K^* = \frac{1}{4} \qquad \text{(B-48)}
$$

B-6 Square-Law Detection of a Sine Wave in Narrow-Band Gaussian Noise

Case I. The assumptions here are those stated in Section B-2, Case I. The probability densities under signal and under no-signal conditions are given by Eqs. (A-20) and (A-21), respectively. Hence

$$
\begin{aligned}
b'(y) &= (y - 1)\, e^{-y}, && y \geq o \\
&= o\,, && y < o
\end{aligned}
\qquad \text{(B-49)}
$$

Thus, the likelihood ratio statistic is

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} \frac{b'(y_i)}{f_o(y_i)} \tag{B-50}$$

$$= \frac{1}{n} \sum_{i=1}^{n} (y_i - 1)$$

The efficacy is

$$K^* = \int (y - 1)^2 f_o(y)\, dv \tag{B-51}$$

$$= 1$$

Case II. The assumptions and discussion in connection with Case II of Section B-4 pertain here also. Utilizing Eqs. (A-20) and (A-21) we obtain the probability densities of the square-law envelope V of the difference waveform z(t). These are

$$g_\theta(v) = e^{-\left[v + \frac{\theta}{2}\right]} I_o\left[2\sqrt{\frac{\theta v}{2}}\right], \qquad v \geq o \tag{B-52}$$

$$= o, \qquad v < o$$

$$f_o(v) = e^{-v}, \qquad v \geq o \tag{B-53}$$

$$= o, \qquad v < o$$

Hence

$$b'(v) = \frac{1}{2} (v - 1)\, f_o(v) \tag{B-54}$$

and the statistic and its efficacy are

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} (v_i - 1) \tag{B-55}$$

$$K^* = \frac{1}{4} \tag{B-56}$$

B-7 Detection of Narrow-Band White Gaussian Signal in Additive Narrow-Band White Gaussian Noise

This problem involves the detection of a stochastic signal in noise; hence, the information on the presence or absence of the signal is carried by the scale parameters of the distributions. In fact, the decision on the presence or absence of signal is based on the difference between the variance under signal and under no-signal conditions. Location parameters such as the mean carry no information. Thus, in the detection of stochastic signals in noise, if the means under signal or under no-signal conditions or under both are unknown or non-stationary, they may be subtracted out of the channel output by capacitive filtering. In this manner the amplitude of the channel output will be the same stationary mean, namely zero, under signal and under no-signal conditions, and a difference between the distributions $G_\theta(y)$ and $F_0(y)$ can be attributed to a difference in variance and, hence, to the signal. In this and the following problems, the channel output will be assumed to have been subjected to capacitive filtering prior to being predetection processed or prior to its examination by the detector.

Use of Eq. (A-24) yields

$$b'(y) = \frac{1}{2} (y^2 - 1)\, f_0(y) \tag{B-57}$$

where $f_o(y)$ is given by Eq. (A-25). Thus, the likelihood ratio statistic is

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} \frac{b'(y_i)}{f_o(y_i)}$$
$$= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} (y_i^2 - 1) \tag{B-58}$$

which is a square-law summing or energy device. The efficacy of the statistic is

$$K^* = \frac{1}{4} \int (y^2 - 1)^2 f_o(y)\, dy \tag{B-59}$$
$$= \frac{1}{2}$$

B-8 Envelope Detection of Narrow-Band White Gaussian Signal in Additive White Gaussian Noise

From Eq. (A-29) we obtain

$$b'(y) = \left(\frac{y^2}{2} - 1 \right) f_o(y) \tag{B-60}$$

thus, the likelihood statistic is

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{y_i^2}{2} - 1 \right) \tag{B-61}$$

which is an energy summing device as in the previous case. The efficacy of the statistic is

$$K^* = \int \left(\frac{y^2}{2} - 1\right)^2 f_o(y)\, dy \qquad \text{(B-62)}$$

that is, twice the efficacy obtained when the energy detector is used without envelope predetection processing.

B-9 Square-Law Detection of Narrow-Band White Gaussian Signal in Additive Narrow-Band White Gaussian Noise

From Eq. (A-34) the quantity b'(y) for this problem is obtained. It is

$$b'(y) = (y - 1)\, f_o(y) \qquad \text{(B-63)}$$

thus, the likelihood statistic is

$$L_n^* = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - 1\right)$$

which is a simple summing device. The efficacy of the statistic is

$$K^* = \int (y - 1)^2 f_o(y)\, dy \qquad \text{(B-64)}$$

the same as that obtained by envelope predetection processing the input waveform and using an energy detector.